

Fourier series

let $f: \mathbb{R} \rightarrow \mathbb{C}$ be con't, such that $f(x+1) = f(x)$, i.e. periodic with period 1.

let $C(\mathbb{T})$ be the family of such functions, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Of special interest are the exponentials $e_n \in C(\mathbb{T})$ def. by

$$e_n(x) := e^{2\pi i n x} \quad ; n \in \mathbb{Z},$$

Note $\cdot (e_n, e_m) = \delta(n-m) = \begin{cases} 1, & \text{if } n=m \\ 0, & \text{else.} \end{cases}$

where $(f, g) = \int_0^1 f(x) \bar{g}(x) dx$.

$\cdot f \in C(\mathbb{T})$ can be identified with $f \in C[0,1]$ st. $f(0) = f(1)$.

Def Let $f \in C(\mathbb{T})$ for $k \in \mathbb{Z}$ define its k -th Fourier coeff.

$$\text{as } \hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} dx$$

Def A trig. pol of degree at most N , is of the form

$$S(x) = \sum_{|k| \leq N} c_k e^{2\pi i k x} = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(2\pi k x) + b_k \sin(2\pi k x)$$

The space of such. trig. pol, is denoted by M_N

Note $M_N = \text{span} \{e_n; -N \leq n \leq N\}$.

Claim Let $f \in M_N$, then $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e_k(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$ (2.1)

Pf: Let $f(x) = \sum_{|k| \leq N} c_k \cdot e_k(x)$, then by orthogonality

$$\hat{f}(l) = \sum_{|k| \leq N} c_k (e_k, e_l) = c_l \quad \text{for } |l| \leq N$$

and $\hat{f}(l) = 0$ for $|l| > N$, □

Question Does the inversion formula remain true for $f \in C(\mathbb{T})$?

Def Let $f \in C(\mathbb{T})$, Then its N -th partial Fourier series is

$$S_N f(x) := \sum_{|k| \leq N} \hat{f}(k) e^{2\pi i k x}$$

Note • Not true for all x

• True for a.e. $x \in \mathbb{T}$ (Carleson '67, Fefferman, ... ← extremely difficult)

Convergence in L^2 , and Plancherel

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Lemma Let $f \in C(\mathbb{T})$ and $S \in M_N$. Then

$$\|f - S\|_{L^2} \geq \|f - S_N f\|_{L^2} \quad (21.2)$$

Pf $f - S = f - S_N f + S_N f - S f$. By def.

$$(S_N f, e_k) = \hat{f}(k) = (f, e_k), \text{ for all } |k| \leq N$$

hence $(f - S_N f, e_k) = 0$, $\forall |k| \leq N \Rightarrow$

$$(f - S_N f, S_N f - S f) = 0$$

thus $\|f - S\|_2^2 = \|f - S_N f\|_2^2 + \|S_N f - S\|_2^2$, (21.3) \square

Note This means $S_N f$ is the orth proj: of f to M_N .

Cor 1 (i) $\|f\|_2^2 \geq \sum_{|k| \leq N} |\hat{f}(k)|^2$ (Bessel's ineq.)

(ii) $\hat{f}(k) \rightarrow 0$ as $|k| \rightarrow \infty$ (Riemann-Lebesgue lemma)

Pf (i) Take $S=0$ in (21.3)

(ii) Follows from (i)

Thm (Plancherel) Let $f \in L^2(\mathbb{T})$, then

(2.6) $\|f\|_2^2 = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^2$, moreover if $g \in L^2(\mathbb{T})$, then

(2.7) $(f, g) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \overline{\hat{g}(k)}$ ($:= (\hat{f}, \hat{g})$)

Pf: First let $f \in C(\mathbb{T})$. Then by (2.3)

$$\|f\|_2^2 = \|f - S_N f\|_2^2 + \|S_N f\|_2^2$$

Let $\epsilon > 0$, by Weierstrass' thm $\exists S \in M_N$ trig. pd.

s.t. $\|f - S\|_{L^2} \leq \|f - S\|_{L^\infty} \leq \epsilon \Rightarrow \|f - S_N\|_{L^2} \leq \epsilon$.

Thus $\|f\|_2^2 \leq \|S_N f\|_2^2 + \epsilon^2 \leq \sum_{|k| \leq N} |\hat{f}(k)|^2 + \epsilon^2$

as $\|f\|_2^2 \geq \sum_{|k| \leq N} |\hat{f}(k)|^2$ (2.7) follows by letting $N \rightarrow \infty$.

If $f \in L^2(\mathbb{T}) \exists g \in C(\mathbb{T})$ s.t. $\|f - g\|_{L^2} \leq \epsilon \Rightarrow$
trig. pol's are also dense in L^2 , thus the previous argument holds.

Use the "polarization identity"

$$(f, g) = \frac{1}{4} \sum_{l=0}^3 (f + i^l g, f + i^l g) i^l$$

for both (f, g) and (\hat{f}, \hat{g}) , to obtain (2.7)

Def Let $n \geq 1$, $C^{(n)}(\mathbb{T}) = \{ f \in C(\mathbb{T}) \text{ such that } f^{(n)} \in C(\mathbb{T}) \}$,
 when $f^{(n)}(x)$ is the n 'th derivative of $f(x)$.

Claim $\widehat{f^{(n)}}(k) = (2\pi i k)^n \widehat{f}(k)$

Pf Induction on n ; and integration by parts for $n=1$.

Note This means that the Fourier transform diagonalizes differential operators, hence its usefulness in ODE & PDE.

Lemma (Chernoff). Let $f \in C^{(1)}(\mathbb{T})$. Then the Fourier series

$$S(x) := \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{2\pi i k x}$$

converges absolutely and uniformly and defines a function

$$S \in C(\mathbb{T}).$$

Pf: First we show: $\sum_{k \in \mathbb{Z}} |\widehat{f}(k)| < \infty$.

We have $|\widehat{f}(k)| \leq C |\widehat{f'}(k)| / |k|$. Thus

$$\sum_{k \neq 0} |\widehat{f}(k)| \leq C \sum_{k \neq 0} |\widehat{f'}(k)| \cdot |k|^{-1} \leq C_1 \sum_{k \neq 0} |\widehat{f'}(k)|^2 \leq C_1 \|f'\|_2^2.$$

This implies that $S_N f(x) = \sum_{|k| \leq N} \hat{f}(k) e^{2\pi i k x}$

converges absolutely and uniformly as $N \rightarrow \infty$, hence

$$S(x) := \lim_{N \rightarrow \infty} S_N f(x) \in C(\mathbb{T})$$

Note As $\widehat{S_N f}(k) = \hat{f}(k)$ for $|k| \leq N$ □

and $S_N f \rightarrow S$ uniformly, we have that

$$\widehat{S}(k) = \hat{f}(k) \text{ for all } k \in \mathbb{Z}$$

that is $(S - f, e_k) = 0 \quad \forall k \in \mathbb{Z} \Rightarrow S = f$

We proved by Weierstrass thm.

Thm Let $f \in C^{(1)}(\mathbb{T})$, then for all x , we have

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$$

, moreover the series converge absolutely and uniformly.

This is a version of the Fourier inversion formula.

Thm (Poisson-summation) let $f \in C^{(1)}(\mathbb{R})$ st. $|f(x)|, |f'(x)| \leq C(1+|x|)^{-1-\delta}$.

Then
$$\sum_{m \in \mathbb{Z}} f(m) = \sum_{k \in \mathbb{Z}} \hat{f}(k)$$

Pf. let $f_p(x) := \sum_{m \in \mathbb{Z}} f(x+m)$, then $f_p \in C^{(1)}(\mathbb{T})$, hence
$$\sum_{m \in \mathbb{Z}} f(m) = f_p(0) = \sum_{k \in \mathbb{Z}} \widehat{f_p}(k) = \sum_{k \in \mathbb{Z}} \hat{f}(k) \quad \square$$