

Spherical maximal thm. - II

For $t > 0$, let $A_t f(x) = f * \sigma_t(x) = \int f(x-y) d\sigma_t(y) = \int f(x-ty) d\sigma(y)$

For $L > 0$, define the max. op :

$$A_{\leq L}^* f(x) = \sup_{0 < t \leq L} |A_t f(x)| \quad \text{and} \quad A_L^* f(x) = \sup_{L \leq t \leq 2L} |A_t f(x)|, \quad (20.1)$$

We have

$$A^* f(x) = \lim_{L \rightarrow \infty} A_{\leq L}^* f(x) = \liminf_{L \rightarrow \infty} A_{\leq L}^* f(x) \quad \xrightarrow{\text{Fatou's Lemma}}$$

$$\|A^* f\|_p \leq \liminf_{L \rightarrow \infty} \|A_{\leq L}^* f\|_p$$

Cor 1 If $\|A_{\leq L}^* f\|_p \leq C_p \|f\|_p$ (with $C > 0$ independent of L)

$$\text{then } \|A^* f\|_p \leq C_p \|f\|_p \quad (20.2)$$

Scaling

Let $L > 0$, $0 \leq t \leq L$ and let $t = L \cdot s$,

For $f: \mathbb{R}^n \rightarrow \mathbb{C}$ write $f_L(x) := f(Lx)$.

$$\begin{aligned} \text{Then } (A_t f)_L(x) &= A_t f(Lx) = \int f(Lx - ty) d\sigma(y) = \\ &= \int f_L(x - sy) d\sigma(y) = (A_s f_L)(x) \end{aligned}$$

$$\Rightarrow (A_{\leq L}^* f)_L = A_{\leq 1}^* f_L \quad \text{and} \quad (A_L^* f)_L = A_1^* f_L \quad (20.3)$$

Since $\|f_L\|_p = L^{-1/p} \|f\|_p$, (20.3) implies that

$$\|A_{\leq L}^* \|_{p \rightarrow p} = \|A_{\leq 1}^* \|_{p \rightarrow p} \text{ and } \|A_L^* \|_{p \rightarrow p} = \|A_1^* \|_{p \rightarrow p}$$

Cor 2 $\|A^* \|_{p \rightarrow p} = \|A_{\leq 1}^* \|_{p \rightarrow p}$ (20.4)

Pf: Clearly $\|A_{\leq 1}^* f\|_p \leq \|A^* f\|_p$.

Also if $\|A_{\leq 1}^* \|_{p \rightarrow p} = C_p$ then $\|A_{\leq L}^* \|_{p \rightarrow p} = C_p$

$$\Rightarrow \|A^* \|_{p \rightarrow p} \leq \liminf_{L \rightarrow \infty} \|A_{\leq L}^* \|_{p \rightarrow p} \leq C_p$$

□

Maximal functions attached to a scale

① The spherical measure $S^{n-1} = \{x; |x|^2=1\}$,

thus formally $\sigma(x) = \delta_0(|x|^2-1) = \int_{\mathbb{R}} e^{2\pi i s (|x|^2-1)} ds$ (20.5)

Lemma Let $f \in S(\mathbb{R}^n)$, $= \lim_{\epsilon \rightarrow 0} \int e^{2\pi i s (|x|^2-1)} \varphi(\epsilon s) ds$

Then $\int f(x) d\sigma(x) = \lim_{\epsilon \rightarrow 0} c_n \iint e^{2\pi i s (|x|^2-1)} f(x) \varphi(\epsilon s) ds dx$

and similarly for $t > 0$ (with $c_n =$) (20.6)

$$\int f(x) d\sigma_t(x) = \lim_{\epsilon \rightarrow 0} c_n t^{-n+2} \iint e^{2\pi i s (|x|^2-t^2)} f(x) \varphi(\epsilon s) ds dx$$

Pf: For $\varepsilon > 0$, then

$$\begin{aligned} \iint e^{2\pi i s(|x|^2-1)} \varphi(\varepsilon s) f(x) dx &= \int \varepsilon^{-1} \hat{\varphi}\left(\frac{|x|^2-1}{\varepsilon}\right) f(x) dx = \\ &= \int_0^\infty \int_{S^{n-1}} \underbrace{\varepsilon^{-1} \hat{\varphi}\left(\frac{r^2-1}{\varepsilon}\right) r^{n-1}}_{\psi_\varepsilon(r^2-1) \cdot r^{n-1}} f(ry) d\sigma(y) dr \quad ; \text{ let } \psi_\varepsilon = \varepsilon^{-1} \hat{\varphi}\left(\frac{\cdot}{\varepsilon}\right) \end{aligned} \quad (20.7)$$

Let $u: r^2-1 \Rightarrow r = (u+1)^{\frac{1}{2}} \Rightarrow \psi_\varepsilon(r^2-1) r^{n-1} dr = \frac{1}{2} \psi_\varepsilon(u) (u+1)^{\frac{n}{2}-1} du$
 $dr = \frac{1}{2}(u+1)^{-\frac{1}{2}}$

$$h(r) := \int_{S^{n-1}} f(ry) d\sigma(y)$$

Thus the expression in (20.7) is

$$\frac{1}{2} \int \psi_\varepsilon(u) (u+1)^{\frac{n}{2}-1} h((u+1)^{\frac{1}{2}}) \longrightarrow \frac{1}{2} h(1) \text{ as } \varepsilon \rightarrow 0$$

as $\{\psi_\varepsilon(u)\}_{\varepsilon \rightarrow 0}$ is an approximation of the identity.

Since $h(1) = \int f(y) d\sigma(y)$ (20.6) follows for $t=1$.

If $t > 0$, then

$$\begin{aligned} \int f(x) d\sigma_t(x) &= \int f(tx) d\sigma(x) = \\ &= c_t \lim_{\varepsilon \rightarrow 0} \iint e^{2\pi i s(|x|^2-1)} f(tx) \varphi(\varepsilon s) dx ds = \end{aligned}$$

$$\text{let } y := tx \Rightarrow dx = t^{-n} dy$$

$$= c \lim_{\varepsilon \rightarrow 0} t^{-n} \iint e^{2\pi i s \left(\frac{|y|^2 - t^2}{t^2} \right)} f(y) \varphi(\varepsilon s) ds dy$$

let $r := st^{-2} \Rightarrow ds = t^2 dr$

$$= c \lim_{\varepsilon \rightarrow 0} t^{-n+2} \iint e^{2\pi i r (|y|^2 - t^2)} f(y) \varphi(\underbrace{\varepsilon t^2 s}_{\varepsilon' \rightarrow 0}) ds dy \Rightarrow (20.6)$$

Cor 1 Let $f \geq 0$. Then for $1 \leq t \leq 2$, one has

$$A_t f(x) \leq c \int_{\mathbb{R}} |f * G_{2-2is}(x)| ds, \text{ where } (20.7)$$

$$G_z(x) = e^{-\pi z |x|^2}, \text{ for } z \in \mathbb{C}, \operatorname{Re} z > 0 \text{ (complex Gaussian)}$$

Proof Since for all $1 \leq t \leq 2$ we have $\operatorname{supp} \sigma_t \subseteq \{1 \leq |y| \leq 2\}$

$$A_t f(x) = \int f(x-y) d\sigma_t(y) \leq c \int f(x-y) e^{-2\pi |y|^2} d\sigma_t(y)$$

Also, for $1 \leq t \leq 2$

$$e^{-2\pi |y|^2} \sigma_t(y) = c_n \lim_{\varepsilon \rightarrow 0} \int e^{2\pi i s |y|^2} e^{-2\pi |y|^2} e^{2\pi i s t^2} \varphi(\varepsilon s) ds$$

$$= c_n \lim_{\varepsilon \rightarrow 0} \int G_{2-2is}(y) e^{2\pi i s t^2} \varphi(\varepsilon s) ds$$

$$\Rightarrow |f * \sigma_t| \leq c_n \int |f * G_{2-2is}| ds \quad \square$$

Cor: $\|A_t^* f\|_p \leq c_n \int \|f * G_{2-2is}\|_p ds \quad (20.8)$

We'll now prove that

$$\|A_n^* f\|_2 \leq C \|f\|_2 \quad \text{for } n \geq 3 \quad \text{which implies the}$$

Spher. Max. Thm for $p=2$.

First we deal with scale $L=1$

Lemma Let $n \geq 3$. Then

(a) $\|A_n^* f\|_2 \leq C_n \|f\|_2$

(b) Suppose $\text{supp } \hat{f} \in [2^k, 2^{k+1}]$, Then

$$\|A_n^* f\|_2 \leq C_n 2^{-ck} \|f\|_2, \text{ for some constant } c > 0.$$

Proof

$$\|f * G_{2-2is}\|_2 \leq \|f\|_2 \|\widehat{G}_{2-2is}\|_\infty$$

We have $|\widehat{G}_z(z)| \leq C_n |z|^{-\frac{n}{2}} e^{-\frac{\pi}{2}|z|^2} \leq C_n |z|^{-\frac{n}{2}}$

(a) $\Rightarrow \|\widehat{G}_{2-2is}\|_\infty \leq C_n (1+|s|)^{-\frac{n}{2}} \Rightarrow$

$$\int \|\widehat{G}_{2-2is}\|_2 ds \leq C_n \int_{\mathbb{R}} (1+|s|)^{-\frac{n}{2}} ds \leq C_n' \text{ (for } n \geq 3)$$

(b) $|e^{-\frac{\pi|z|^2}{2-2is}}| = e^{-\frac{\pi|z|^2}{2(1+s^2)}} \leq e^{-\frac{\pi k^2}{2(1+s^2)}} \text{ for } k \leq |z| \leq 2k.$

Thus if $\text{supp } \hat{f} \in [k, 2k]$ with $k=2^k$,

$$\|f * G_{2^{-2}is}\|_2^2 = \int_{K \leq |z| \leq 2K} |\hat{f}(z)|^2 |\hat{G}_{2^{-2}is}(z)|^2 \leq C_n (1+|s|)^{-n} e^{-\frac{\pi K^2}{1+s^2}} \|f\|_2^2$$

$$\Rightarrow \int_{\mathbb{R}} \|f * G_{2^{-2}is}\|_2 ds \leq C_n \|f\|_2 \int_{\mathbb{R}} (1+|s|)^{-\frac{n}{2}} e^{-\frac{\pi K^2}{2(1+s^2)}} ds \leq C_n K^{-\frac{n}{4}} \|f\|_2$$

By scaling, this implies more generally

$$\|A_{2^{-k}}^* f\|_2 \leq C_n 2^{-cl} \|f\|_2 \quad \text{is} \quad \text{supp } \hat{f} \subseteq [2^{k+l}, 2^{k+l+1}], \quad l \geq 0.$$

Indeed let $L := 2^{-k}$ then we have

$$(A_L^* f)_L = A_1^* f_L \quad ; \quad \text{supp } \hat{f}_L = \text{supp } \hat{f}(2^k z) \subseteq [2^l, 2^{l+1}]$$

$$\text{(as } f_L(x) = f_L(x) \Rightarrow \hat{f}_L(z) = L^{-n} \hat{f}(L^{-1}z))$$

$$\text{Thus } \|A_1^* f_L\|_2 \leq C_n 2^{-cl} \|f_L\|_2 \Rightarrow (2.10)$$

Orthogonality and the "full" spherical maximal thm, for $p=2$. □

Let $\varphi_0 \in C_0^\infty(\mathbb{R})$, s.t. $0 \leq \varphi_0 \leq 1$, and $\varphi_0(x) = 1$ for $|x| \leq 1$.

$$\text{Then } 1 = \varphi_0(z) + \sum_{j \geq 1} \varphi_0(2^{-j}z) - \varphi_0(2^{-(j+1)}z) = \varphi_0(z) + \sum_{j \geq 1} \varphi_1(2^{-j}z)$$

with

$$\varphi_1(z) = \varphi\left(\frac{z}{2}\right) - \varphi\left(\frac{z}{4}\right). \quad \text{Note, that: } \text{supp } \varphi_1(2^{-j}z) \subseteq [2^j, 2^{j+1}]$$

Accordingly, write

$$\hat{f}(z) = \hat{f}(z)\psi_0(z) + \sum_{j \geq 1} \hat{f}(z)\psi_j(z) \quad \text{with } \psi_j(z) = \psi_1(2^{-j}z)$$

and hence

$$f = f_0 + \sum_{j \geq 1} f_j, \quad f_0 = f * \psi_0, \quad f_j = f * \psi_j, \quad \psi_j = \psi_j^\vee.$$

Now

$$A_{\leq 1}^* f = \sup_{k \geq 0} A_{2^{-k}}^\vee f = \sup_{k \geq 0} A_{2^{-k}}^* (g_k + \sum_{l=1}^{\infty} f_{k+l})$$

where

$$g_k = f_0 + f_1 + \dots + f_k$$

First,

$$\hat{g}_k = \hat{f}(\psi_0 + \psi_1 + \dots + \psi_k) \Rightarrow \hat{g}_k(z) = \hat{f}(z) \hat{\psi}_0(2^{-k}z)$$

$$\Rightarrow g_k = f * \psi_{0,k} \quad \text{with } \psi_{0,k}(x) = 2^{+nk} \psi_0(2^{-k}x)$$

Claim:

$$A_{2^{-k}}^* g_k \leq C Mf, \quad \text{where } M \text{ is the standard maximal function associated to balls}$$

Pf

Heuristics:

$\psi_0 \in S(\mathbb{R}^n)$ quickly decreasing thus

$\psi_{0,k}(x)$ is quickly decreasing and is negligible for

$|x| \gg 2^{-k}$. If $2^{-k} \leq t \leq 2^{-k+1}$ then the

same holds for $\Psi_{0,k} * \sigma_t$, uniformly for t .

Thus $A_t g_k = g_k * \sigma_t = f * (\Psi_{0,k} * \sigma_t) \leq C$ for $\tilde{\Psi}_k$
 for all $2^{-k} \leq t \leq 2^{-k+1}$ where $\tilde{\Psi}_k(x) = 2^{nk} \tilde{\Psi}(2^k x)$
 for some $\tilde{\Psi}$ quickly decreasing.

It is then enough to show that

$f * \tilde{\Psi}_k(x) \leq C Mf(x)$, but $f * \tilde{\Psi}_k$ is essentially
 an average of f at scale $2^{-k} \Rightarrow$ bounded by $Mf(x)$

For the precise argument one needs to show. □

• $\sup_{2^{-k} \leq t \leq 2^{-k+1}} |\Psi_{0,k} * \sigma_t(x)| \leq C \tilde{\Psi}_k(x) = C 2^{nk} \tilde{\Psi}(2^k x)$

for a function $\tilde{\Psi}(x) \leq C_n (1 + |x|)^{-n-1}$ (say).

• $f * \tilde{\Psi}_k(x) \leq C Mf(x)$ (by decomposing $|x|$ dyadically
 and accounting for the "tails")

Orthogonality arguments

So

$$\begin{aligned} \|A_{\leq 1} f\| &\leq \sup_{k \geq 0} \left(A_{2^{-k}}^* g_k + A_{2^{-k}}^* \left(\sum_{\ell=1}^{\infty} f_{k+\ell} \right) \right) \\ &\leq C Mf + \sup_{k \geq 0} A_{2^{-k}}^* \left(\sum_{\ell=1}^{\infty} f_{k+\ell} \right) \end{aligned}$$

Also, as $A_{2^{-k}}^*(f+g) = A_{2^{-k}}^*f + A_{2^{-k}}^*g$;

$$\sup_{k \geq 0} A_{2^{-k}}^* \left(\sum_{\ell=1}^{\infty} f_{k+\ell} \right) \leq \left[\sum_{k \geq 0} \left(\sum_{\ell \geq 1} A_{2^{-k}}^* f_{k+\ell} \right)^2 \right]^{\frac{1}{2}} \quad (20.11)$$

Let $g_\ell(k) = A_{2^{-k}}^* f_{k+\ell}$; then the R.H.S of (20.11) can be estimated via Minkowski's ineq. as :

$$\begin{aligned} \left[\sum_{k \geq 0} \left(\sum_{\ell \geq 1} g_\ell(k) \right)^2 \right]^{\frac{1}{2}} &= \left\| \sum_{\ell \geq 1} g_\ell \right\|_{\ell^2} \leq \sum_{\ell \geq 1} \|g_\ell\|_{\ell^2} = \\ &= \sum_{\ell \geq 1} \left(\sum_{k \geq 0} |A_{2^{-k}}^* f_{k+\ell}|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Now, let $h_\ell(x) = \left(\sum_{k \geq 0} |A_{2^{-k}}^* f_{k+\ell}(x)|^2 \right)^{\frac{1}{2}}$, then we have

so far

$$\begin{aligned} \left\| \sup_{k \geq 0} A_{2^{-k}}^* \left(\sum_{\ell \geq 1} f_{k+\ell}(x) \right) \right\|_2 &\leq \left\| \sum_{\ell \geq 1} h_\ell(x) \right\|_2 \leq \\ &\leq \sum_{\ell \geq 1} \|h_\ell(x)\|_2 = \sum_{\ell \geq 1} \left\| \left(\sum_{k \geq 0} |A_{2^{-k}}^* f_{k+\ell}(x)|^2 \right)^{\frac{1}{2}} \right\|_2 = \\ &= \sum_{\ell \geq 1} \left(\int_{\mathbb{R}^n} \sum_{k \geq 0} |A_{2^{-k}}^* f_{k+\ell}(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \sum_{\ell \geq 1} \left(\sum_{k \geq 0} \|A_{2^{-k}}^* f_{k+\ell}\|_2^2 \right)^{\frac{1}{2}} \end{aligned}$$

Summarizing we have

$$\|A_{\leq 1}^* f\|_2 \leq c \|Mf\|_2 + \sum_{\ell \geq 1} \left(\sum_{k \geq 0} \|A_{2^{-k}}^* f_{k+\ell}\|_2^2 \right)^{\frac{1}{2}} \quad (20.12)$$

By (20.) we have

$$\|A_{2^{-\ell}}^* f_{k+\ell}\|_2^2 \leq 2^{-c\ell} \|f\|_2^2, \text{ thus}$$

$$\sum_{k \geq 0} \|A_{2^{-k}}^* f_{k+\ell}\|_2^2 \leq 2^{-c\ell} \sum_{k \geq 0} \|f_{k+\ell}\|_2^2 \leq C_n 2^{-c\ell} \|f\|_2^2$$

Indeed:

$$\sum_{k \geq 0} \|f_{k+\ell}\|_2^2 = \int |\hat{f}(\xi)|^2 \left(\sum_{k \geq 0} \varphi_{k+\ell}^2(\xi) \right) \leq c \int |\hat{f}(\xi)|^2$$

Finally as $\|Mf\|_2 = c \|f\|_2$, we have ≤ 1

$$\|A_{\leq 1}^* f\|_2 \leq c \|f\|_2 + \left(\sum_{\ell \geq 1} 2^{-c\ell} \right) \|f\|_2 \leq c \|f\|_2. \quad \square$$