

Maximal functions

Let B_r be the ball of radius r centered at 0, and let $|B_r|$ denote the volume of B_r .

Note We have the following properties

- (1) $B(x, r) \cap B(y, r) \neq \emptyset \Rightarrow B(y, r) \subseteq B(x, c_1 r)$ ($c = 3$ e.g.)
- (2) $|B(x, c_1 r)| \leq c_2 |B(x, r)|$ ($\forall c_1 > 0 \exists c_2 > 0$)

Define the maximal function $Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \left| \int_{B_r} f(x-y) dy \right|$

Thm For $1 \leq p \leq \infty$, $\|Mf\|_{L^p(\mathbb{R}^n)} \leq c_p \|f\|_{L^p(\mathbb{R}^n)}$

Covering Lemma (Vitaly) Let $E \subseteq \mathbb{R}^n$ be a measurable set

which is covered by a finite collection of balls $\{B_j\}$

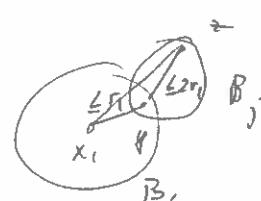
Then there is a disjoint subcollection of balls B_1, \dots, B_m

s.t. $\sum_{k=1}^m |B_k| \geq c |E|$

Pf Let $B_i = B_i(x_i, r_i)$ be a ball of maximal radius. If

$B_j \cap B_i \neq \emptyset$ then $B_j \subseteq B(x_i, c_1 r_i) = B_i^*$

- Keep B_i and delete all B_j s.t. $B_j \cap B_i \neq \emptyset$
Note, that the rest of the balls are disjoint from B_i



- Select $B_2 = B_2(x_2, r_2)$ of max radius from the remaining balls and repeat the procedure until no balls left.

This gives B_1, \dots, B_m disjoint so that $E \subseteq B_1^* \cup \dots \cup B_m^*$

$$|E| \leq \sum_{i=1}^m |B_i^*| \leq C_2 \sum_{i=1}^m |B_i| \quad \square$$

Lemma 1. Let $f \in L^1(\mathbb{R}^n)$, Then $\forall \alpha > 0$, we have

$$|\{x : Mf(x) > \alpha\}| \leq \frac{c}{\alpha} \|f\|_1 \quad (2)$$

Note If $g \in L^1$ then

$$\|g\|_1 \geq \int_{\{|g(x)| > \alpha\}} |g(x)| \geq \alpha |\{x : |g(x)| > \alpha\}| \Rightarrow |\{x : |g(x)| > \alpha\}| \leq \frac{\|g\|_1}{\alpha}$$

thus (2) is called a weak 1-1 inequality, i.e.

Mf is in weak- L^1

Pf Let $E_\alpha = \{Mf(x) > \alpha\}$, let $E \subseteq E_\alpha$ compact. If

$$x \in E \text{ then } \exists B_x \text{ s.t. } \int_{B_x} |f(x)| dx > \alpha |B_x|. \quad (3)$$

Since $E \subseteq \bigcup_{x \in E} B_x$ we have E is covered by finitely many of these balls, thus $\exists B_1, \dots, B_m$ disjoint balls, s.t.

$$\forall 1 \leq i \leq m \quad \int_{B_i} |f(x)| > \alpha |B_i| \quad (3)'$$

$$\text{and} \quad |E| \leq c_n \sum_{i=1}^m |B_i| \quad (c_n = 3^n) \quad (4)$$

$$\text{thus} \quad |E| \leq c_n \sum_i |B_i| \leq \frac{c_n}{\alpha} \sum_{i=1}^m \int_{B_i} |f(x)| \leq \frac{c_n}{\alpha} \|f\|_1.$$

Pf of Thm 1. Note that $\|Mf\|_\infty \leq \|f\|_\infty$. □

$$\text{Now } \|Mf\|_p^p = \int |Mf(x)|^p dx = p \int_0^\infty |\{Mf > \alpha\}| \alpha^{p-1} d\alpha$$

let $f_1(x) = f(x)$ if $|f(x)| \geq \frac{\alpha}{2}$ and $f_1(x) = 0$ if $|f(x)| \leq \frac{\alpha}{2}$,

then $|f(x)| \leq |f_1(x)| + \frac{\alpha}{2}$ and hence $Mf \leq Mf_1 + \frac{\alpha}{2}$

$$\Rightarrow \{Mf > \alpha\} \subseteq \left\{Mf_1 > \frac{\alpha}{2}\right\}.$$

Now

$$\begin{aligned} \|Mf\|_p^p &= p \int_0^\infty |\{Mf > \alpha\}| \alpha^{p-1} d\alpha \leq p \int_0^\infty |\{Mf_1 > \frac{\alpha}{2}\}| \alpha^{p-1} d\alpha \\ &\leq C_p \int_0^\infty \int_{\mathbb{R}^n} |f_1(x)| \alpha^{p-2} dx = C_p \int_0^\infty \int_{\mathbb{R}^n} |f(x)| \mathbf{1}_{\{\frac{\alpha}{2} \leq |f(x)|\}} \alpha^{p-2} dx \\ &= C_p \int_{\mathbb{R}^n} |f(x)| \int_0^\infty \alpha^{p-2} d\alpha = \frac{C_p}{p-1} \int_{\mathbb{R}^n} |f(x)|^p dx = \frac{C_p}{p-1} \|f\|_p^p \end{aligned}$$

□

Corollary (Lebesgue density thm)

Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$ we have

$$f(x) = \lim_{r \rightarrow 0} \frac{1}{|B_r|} \int_{x+B_r} f(y) dy \quad (5)$$

Pf: If $f \in C(\mathbb{R}^n)$ then (5) holds. Fix $R > 1, \alpha > 0$.

$$\text{Let } f_r(x) := \frac{1}{|B_r|} \int_{x+B_r} f(y) dy, \text{ let } \varepsilon > 0.$$

If $f \in L^1_{loc}(\mathbb{R})$ then $\exists g \in C(\mathbb{R})$ s.t. $\|f - g\|_{L^1(B_R)} \leq \varepsilon$

$$\text{Write } f = g + h \Rightarrow f_r = g_r + h_r \Rightarrow f - f_r = g - g_r + h - h_r$$

$$\text{Now } \limsup_{r \rightarrow 0} |f - f_r| \leq \limsup_{r \rightarrow 0} |g - g_r| + \sup_{r \rightarrow 0} |h - h_r| = M(h - h_r)$$

$$\text{Thus } |\{ \limsup_{r \rightarrow 0} |f - f_r| > \alpha \}| \leq |\{|h| > \frac{\alpha}{2}\}| + |\{Mh > \frac{\alpha}{2}\}| \leq |h| + M|h|$$

$$\text{Letting } \varepsilon \rightarrow 0 \text{ gives } \limsup_{r \rightarrow 0} |f - f_r| = 0 \text{ for a.e. } x \in B_R$$

$$\Rightarrow f_r(x) \rightarrow f(x) \text{ as } r \rightarrow 0 \text{ for a.e. } x \in B_R.$$

Further examples

$$B_r = \{x_i \mid |x_i| \leq r^i\} \subseteq \mathbb{R}^n$$

- Let $r \circ x = (rx_1, \dots, r^n x_n)$ (non-isotropic dilation)

Then $x \circ B_r = r \circ B_1 = \{r \circ x \mid x \in B_1\}$. Let $B_{x,r} = x + B_r$

- The family $\{B_{x,r}\}_{r>0}$ satisfies properties (i) & (ii).

Indeed $|B_r| = r^{\frac{n(n+1)}{2}}$, thus $|B_{2r}| \leq c_n |B_r|$ ($c_n = 2^{\frac{n(n+1)}{2}}$)

$$\text{and in general } |B_{cr}| \leq c^{\frac{n(n+1)}{2}} |B_r| \quad (1)$$

If $z \in B_{x,r} \cap B_{y,s} \neq 0$ and $r \geq s$, then

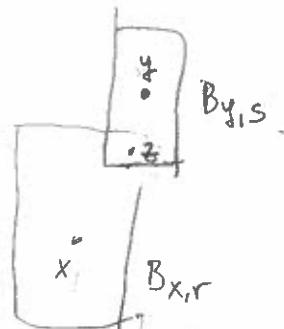
$$|x_i - z| \leq r^i, \quad |y_i - z| \leq s^i \leq r^i$$

$$\Rightarrow |x_i - y_i| \leq 2r^i \leq (2r)^i \quad (1 \leq i \leq n)$$

$$\Rightarrow z \in B_{x,r}$$

$$\text{If } u \in B_{y,s} \text{ then } |u_i - y_i| \leq s^i \leq r^i \Rightarrow |u_i - x_i| \leq 3^i + r^i$$

$$\Rightarrow u \in B_{x,3r} \Rightarrow B_{y,s} \subseteq B_{x,3r}.$$



Thus the maximal theorem holds; i.e. For the maximal operator:

$$A^*f(x) = \sup_{r>0} \frac{1}{|B_r|} \left| \int_{B_r} f(x-y) dy \right|$$

$$\text{we have } \|A^*f\|_P \leq c_p \|f\|_P$$

Corollary (Density theorem) Let $f \in L^p$ ($1 < p < \infty$).

For a.e. $x \in \mathbb{R}^n$, we have $\lim_{r \rightarrow 0} A_r f(x) = f(x)$

$$\text{where } A_r f(x) = \frac{1}{|B_r|} \int_{x+B_r} f(y) dy$$

Pf. It is enough to show that $\overline{\lim}_{r \rightarrow 0} |A_r f(x) - f(x)| = 0$

$$\text{for a.e. } x \Leftrightarrow \left\| \overline{\lim}_{r \rightarrow 0} |A_r f(x) - f(x)| \right\|_p = 0,$$

Let $\varepsilon > 0$ and write $f = g + h$ with $g \in C(\mathbb{R}^n)$ (cont.)

$$\text{Then } |A_r f - f| \leq |A_r g - g| + |A_r h - h| \quad \text{and } \|h\|_p \leq \varepsilon.$$

$$\Rightarrow \overline{\lim}_{r \rightarrow 0} |A_r f - f| \leq \overline{\lim}_{r \rightarrow 0} |A_r g - g| + \overline{\lim}_{r \rightarrow 0} |A_r h - h|$$

Since g is cont; $\forall y > 0 \exists r_0$ s.t. if $|y| \leq r_0$ then

$$|f(x-y) - f(x)| \leq \gamma \Rightarrow |A_r f(x-y) - f(x)| =$$

$$= \frac{1}{|B_r|} \left| \int_{B_r} (g(x-y) - g(x)) dy \right| \leq \frac{1}{|B_r|} \int_{B_r} \gamma \leq \gamma \quad \text{if } r \leq r_0,$$

Thus $|A_r g(x) - g(x)| \rightarrow 0$ as $r \rightarrow 0 \Rightarrow \overline{\lim}_{r \rightarrow 0} |A_r g - g| = 0$.

Then $\overline{\lim}_{r \rightarrow 0} |A_r f - f| \leq \sup_{r > 0} |A_r h - h| \leq A_h + h$

Taking the L^p -norm of both sides

$$\left\| \lim_{r \rightarrow 0} |Arf - f| \right\|_{L^p} \leq \|A_\infty h\|_{L^p} + \|h\|_{L^p} \leq (c_p + 1) \|h\|_p \leq (c_p + 1) \varepsilon$$

Since this hold for all $\varepsilon > 0$, we get $\lim_{r \rightarrow 0} |Arf - f| = 0$

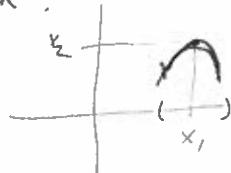
Maximal functions & averages over curves for a.e. $x \in \mathbb{R}^n$ \square

- Let $\gamma(t) = (t, t^2) : \mathbb{R} \rightarrow \mathbb{R}^2$ ("parabola")

For $r > 0$, define $Arf(x_1, x_2) = \frac{1}{2r} \int_{-r}^r f(x_1 - t, x_2 - t^2) dt$

Q: Is it true that $\lim_{r \rightarrow 0} Arf(x) = f(x)$ for a.e. $x \in \mathbb{R}^2$?

Let $A^*f(x) = \sup_{r>0} |Arf(x)|$.



Thm For $1 < p \leq \infty$, we have $\|A^*f\|_{L^p} \leq c_p \|f\|_{L^p}$.

Note • WLOG can assume $f \geq 0$, as $|A^*f| \leq A^*(|f|)$ and $\|f\|_p = \||f|\|_p$.

- If $2^{j-1} \leq r \leq 2^j$ and $f \geq 0$, then

$$Arf(x) \leq \frac{1}{2^j} \int_{-2^j}^{2^j} f(x_1 - t, x_2 - t^2) dt \leq 2A_{2^j}f(x)$$

Thus $A^*f(x) \leq 2 A_0^*f(x) := 2 \sup_{j \in \mathbb{Z}} A_{2^j}f(x)$

Clea: Approximate v_{2j} on the parabola

with $\chi_{2j} = |\mathbb{B}_{2j}|^{-1} \mathbb{B}_{2j}$, where

γ_j is the measure so that

and

$$\mathbb{B}_{2j} = \{(x_1, x_2); |x_1| \leq 2^j, |x_2| \leq 2^{2j}\} \quad (\text{non-isotropic rectangle})$$

Now if $B_D^* f(x) := \sup_{j \in \mathbb{Z}} |f * \chi_{2j}(x)|$ and note that

$$A_D^* f(x) = \sup_{j \in \mathbb{Z}} |f * \gamma_j(x)|$$

$$(\text{as } f * \gamma_r(x) = \int f(x-y) d\gamma_r(y) = \frac{1}{2r} \int_{-r}^r f(x-t, x+t) dt)$$

then

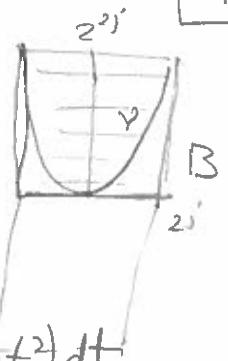
$$|A_{2j}^* f(x)| \leq |A_{2j} f(x) - B_{2j} f(x)| + |B_{2j} f(x)|$$

$$\Rightarrow |A_D^* f(x)| \leq |B_D^* f(x)| + \sup_{j \in \mathbb{Z}} |A_{2j} f(x) - B_{2j} f(x)|$$

$$\Rightarrow |A_D^* f(x)|^p \leq |B_D^* f(x)|^p + \sum_{j \in \mathbb{Z}} |A_{2j} f(x) - B_{2j} f(x)|^p$$

~~$\Rightarrow \|A_D^* f\|_p^p \leq \|B_D^* f\|_p^p + \sum_{j \in \mathbb{Z}} \|A_{2j} f - B_{2j} f\|_p^p$~~

Note $\|B_D^* f\|_p \leq C_p \|f\|_p$.



Lemma Let $p=2$. Then

$$\sum_{j \in \mathbb{Z}} \|A_{2j}f - B_{2j}f\|_{L^2}^2 \leq C \|f\|_{L^2}^2,$$

Pf: $\widehat{A_{2j}f}(3) = \widehat{f * \mathcal{V}_{2j}}(3) = \widehat{f}(3) \widehat{\mathcal{V}_{2j}}(3),$

$$\widehat{\mathcal{V}_r}(3) = \frac{1}{2r} \int_{-r}^r e^{-2\pi i (t3_1 + t^2 3_2)} dt = \int e^{-2\pi i (t3_1 + t^2 3_2)} \chi_r(t) dt =$$

with $\chi_r(t) = \frac{1}{2r} \chi\left(\frac{t}{r}\right); \quad \chi(t) = \mathbb{1}_{[-1, 1]}$.

$$\begin{aligned} \widehat{\mathcal{V}_r}(3) &= \int e^{-2\pi i (t^2 r^2 3_2 + tr 3_1)} \chi(t) dt \quad (\text{by writing } t = \frac{t}{r}) \\ &= \widehat{\mathcal{V}_1}(r_0 3) \end{aligned}$$

So consider

$$\widehat{\mathcal{V}_1}(3_1, 3_2) = \int e^{-2\pi i (t3_1 + t^2 3_2)} \chi(t) dt$$

- If $3_1 = 0$, then $|\widehat{\mathcal{V}_1}(0, 3_2)| \leq C |3_2|^{-\frac{1}{2}}$ as it is an oscillatory integral
- If $3_2 = 0$, then $|\widehat{\mathcal{V}_1}(3_1, 0)| = |\widehat{\chi}(3_1)| \leq C |3_1|^{-1}$