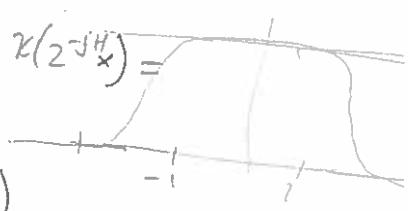


Partition of unity (Supplement to Lecture 16)

Let $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ s.t. $\chi_0(x) = 1$ if $|x| \leq 1$, $\chi_0(x) = 0$ if $|x| \geq 2$,
and let $\chi_j(x) = \chi(2^{-j}x)$ and $\psi_j(x) = \chi(2^{-j}x) - \chi(2^{-j+1}x) =$

then $\chi_0 + \sum_{j=1}^k \psi_j(x) = \chi(2^{-k}x) = 1$ if $k \geq k(x)$



$\Rightarrow \forall x \quad \chi_0(x) + \sum_{j=1}^{\infty} \psi_j(2^{-j}x) = 1$ note $\text{supp } \psi \subseteq \left\{ \frac{1}{2} \leq |x| \leq 2 \right\}$

Let

$$d\sigma = \chi_0 d\sigma + \sum_{j \geq 1} \psi_j d\sigma = \sigma_0 + \sum_{j \geq 1} \sigma_j$$

Estimates we used

$$\widehat{f * d\sigma} = \widehat{f * d\sigma_0} + \sum_{j \geq 1} \widehat{f * d\sigma_j} \xrightarrow{L_q \rightarrow L_q}$$

- $\|\widehat{f * d\sigma_j}\|_\infty \lesssim \|\widehat{d\sigma_j}\|_\infty \|f\|_1 \lesssim 2^{-j\frac{n-1}{2}} \|f\|_1 \quad (i)$

- $\|\widehat{f * d\sigma_j}\|_2 \lesssim \|(\widehat{\psi_j * d\sigma})^\vee\|_\infty \|f\|_2$

$$\|(\widehat{\psi_j * d\sigma})^\vee\|_\infty = \|\widehat{\psi_j * d\sigma}\|_\infty \lesssim 2^j$$

i) This estimate is only based on the fact that

$$\sigma(B(x, r)) \leq C r^{n-1}$$

i.e. that σ is supported on a surface of dim = $n-1$

) Is based on the fact that S^{n-1} has non-vanishing Gaussian curvature.

Lemma (Schur's test) Let (X, μ) , (Y, ν) be meas. spaces, and let $K(x, y)$ be a meas. function on $X \times Y$.

If $\int_X |K(x, y)| d\mu(x) \leq A$ and $\int_Y |K(x, y)| d\nu(y) \leq B$

then one has the bound : $\|Tf\|_{L^2(d\mu)} \leq \sqrt{AB} \|f\|_{L^2(d\nu)}$ (1)

for $Tf(x) = \int_K K(x, y) f(y) d\nu(y)$.

Pf: Note $\forall \varepsilon > 0 : \sqrt{ab} \leq \frac{1}{2}(ea + \varepsilon^{-1}b)$ ($a, b > 0$).

WLOG assume $\|f\|_{L^2(d\nu)} = 1$ and then it is enough

to prove (2) $\int |K(x, y) f(y) g(x)| d\nu(y) d\mu(x) \leq 1$, for all $\|g\|_{L^2(d\mu)} \leq 1$

Now

$$|f(y)| |g(x)| \leq \frac{1}{2} (\varepsilon |f(x)|^2 + \varepsilon^{-1} |g(x)|^2), \text{ thus}$$

$$\begin{aligned} \text{LHS of (2)} &\leq \frac{1}{2} \varepsilon \underbrace{\int |K(x, y)|^2 |f(y)|^2 d\mu(x) d\nu(y)}_{\varepsilon A = A^{\frac{1}{2}} B^{\frac{1}{2}}} + \\ &\quad + \frac{1}{2} \varepsilon^{-1} \int |K(x, y)| |g(x)|^2 d\mu(x) d\nu(y) \end{aligned}$$

$$\leq \frac{1}{2} \varepsilon A \|f\|_{L^2(d\nu)}^2 + \frac{1}{2} \varepsilon^{-1} B \|f\|_{L^2(d\mu)}^2$$

$$\leq \frac{1}{2} (\varepsilon A + \varepsilon^{-1} B) = \sqrt{AB} \quad \text{if } \varepsilon = \frac{\sqrt{B}}{\sqrt{A}}$$

□

Thm Let ν be a positive meas, s.t.

$$\nu(B(x, r)) \leq C r^\alpha$$

Then

$$\|\widehat{f d\nu}\|_{L^2(B_R)} \leq CR^{n-\alpha} \|f\|_{L^2(d\nu)}$$

Pf

Let ϕ be such that $\phi(x) \geq 1$ if $|x| \leq 1$, $\phi(x) = \phi(-x)$
and $\widehat{\phi} = C_0^\infty(\mathbb{R}^n)$. (Let $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi \geq 0$ and $\psi_1 = \psi + \psi$
 $\Rightarrow \widehat{\psi * \psi} = |\widehat{\psi}(x)|^2 = \widehat{\phi} \Rightarrow \phi \geq 0, \phi(0) \geq 0$)

Let $\Phi_R(x) = \phi(R^{-1}x)$, and hence $\phi(x) \geq 1$ for $|x| \leq R \Rightarrow \phi\left(\frac{x}{R}\right) \geq 1$
for $|x| \leq 1$

$$\begin{aligned} \|\widehat{f d\nu}\|_{L^2(B_R)} &\leq \|\Phi_R^{-1} \cdot \widehat{f d\nu}\|_{L^2(dx)} = \\ &= \|\widehat{\Phi_R^{-1} * (f d\nu)}\|_{L^2} \end{aligned}$$

By expanding, $\widehat{\Phi_R^{-1}}(x) = R^n \widehat{\phi}(Rx)$

$$\widehat{\Phi_R^{-1} * (f d\nu)}(x) = \int R^n \phi(R(x-y)) f(y) d\nu(y) = \int K_R(x, y) f(y) d\nu(y)$$

$$\int |K_R(x, y)| dx = R^n \int \widehat{\phi}(R(x-y)) dx = \int \widehat{\phi}(u) dy = \|\widehat{\phi}\|_1 = A$$

$$\int |K_R(x, y)| d\nu(y) = R^n \int |\widehat{\phi}(R(x-y))| dy \leq R^n \nu(B(x, R)) \leq C' R^{n-\alpha}$$

$R|x-y| \in \mathbb{C}$
 $\Rightarrow |x-y| \leq CR^{-1}$

Thus, by Schur's test

$$\|\widehat{\Phi_R^{-1} * (f d\nu)}\|_{L^2} \leq C R^{\frac{n-\alpha}{2}} \|f\|_{L^2(d\nu)}$$

Note This is an $L^2 \rightarrow L^2(B_r)$ type estimate; uses only the dimension

Oscillatory integral operators

Let $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and for $\lambda > 0$, define

$$T_\lambda f(x) = \int e^{-\pi i \lambda \phi(x,y)} a(x,y) f(y) dy$$

Note. The Fourier transform is of "this type" with $a(x,y)=1$ and $\phi(x,y)=x \cdot y$.

- By Schur's test $|T_\lambda f(x)| \leq \int |a(x,y)| |f(y)| dy$
 $\Rightarrow \|T_\lambda f\|_{L^2} \leq C \|f\|_{L^2}$
 but if $\phi(x,y)$ is suff. non-deg., then $\|T_\lambda f\|_{L^2} \leq C \lambda^{-\frac{n}{2}} \|f\|_{L^2}$ also holds.
- If $\phi(x,y) = \phi(x)$ then $|T_\lambda f(x)|$ is independent of λ
 & if $\phi(x,y) = \phi(y)$, then $T_\lambda f(x) = \int a(x,y) f(y) dy$
 (with $|f_\lambda(y)| = |e^{-\pi i \lambda \phi(y)} f(y)| = |f(y)|$) thus there is no decay in λ .

Hence the correct non-deg. condition is:

$$\det(\tilde{H}_\phi(x,y)) = \det\left(\frac{\partial^2 \phi}{\partial x_i \partial y_j}\right)(x,y) \neq 0 \quad \forall (x,y) \in \text{supp } a(x,y)$$

Thm (Hörmander)

If $\phi(x, y)$ is non-deg. on the support of a ,

then $\|T_\lambda f\|_{L^2} \leq C \lambda^{-\frac{n}{2}} \|f\|_{L^2}$.

Pf

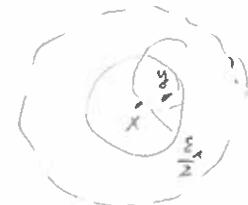
$$T_\lambda^* g(y) = \int e^{-\pi i \lambda} \phi(x, y) a(x, y) g(x) dx; T_\lambda f(x) = \int K_\lambda(x, y) f(y) dy$$

$$\Rightarrow T_\lambda T_\lambda^* f(x) = \int H_\lambda(x, y) |f(y)| dy \quad T_\lambda^* f(y) = \int \bar{K}_\lambda(y, x) f(x) dx$$

with

$$H_\lambda(x, y) = \int K_\lambda(x, z) \bar{K}_\lambda(y, z) dz =$$

$$= \int e^{-\pi i \lambda} [\phi(x, z) - \phi(y, z)] a(x, z) \bar{a}(y, z) dz$$



Now $x \rightarrow \nabla_z \phi(x, z)$ is injective in a ngh of y ; that is $= \int_0^1 \partial_z \phi(y + t(x-y)) - \partial_z \phi(x) dt$

$\forall y \exists \varepsilon_y > 0$ s.t. $\nabla_z \phi(x, z) \neq \nabla_z \phi(y, z)$ if $x \in B(y, \varepsilon_y)$, $x \neq y$.

By compactness $\exists \varepsilon > 0$ (why?) s.t. if $0 < |x-y| < \varepsilon$ then $\nabla_z \phi(x, z) \neq \nabla_z \phi(y, z)$.

Write $a = \sum_{j=1}^J a_j$ s.t. $a_j(x, y) \neq 0 \Rightarrow |y-x| < \frac{\varepsilon}{2} \Rightarrow T_\lambda = \sum_{j=1}^J T_{j, \lambda}$

\Rightarrow thus $T_{j, \lambda} \cdot T_{k, \lambda}^*$ has a kernel of the form

$$\int e^{-\pi i \lambda} (\phi(x, z) - \phi(y, z)) a_j(x, z) \bar{a}_k(y, z) dz$$

Now if $\exists z$: $a_j(x, z) \bar{a}_k(y, z) \neq 0 \Rightarrow |x-z| < \varepsilon/2, |y-z| < \varepsilon/2$

$\Rightarrow |x-y| < \varepsilon \Rightarrow z \rightarrow \phi(x, z) - \phi(y, z)$ has no

critical point on the support of the amplitude.

$\Rightarrow |H_\lambda(x, y)| \leq C_1 (1 + |x-y|)^{-N} \text{ as } |\phi(x, z) - \phi(y, z)| > \dots$