

Restriction problem

let $f: S^{n-1} \mapsto \mathbb{C}$, and consider

$$\widehat{f d\sigma}(\xi) = \int_{S^{n-1}} f(x) e^{-2\pi i x \cdot \xi} d\sigma(x)$$

If $f \in C^p(S^{n-1})$, then we have seen that

$$|\widehat{f d\sigma}(\xi)| \leq C_f |\xi|^{-\frac{n-1}{2}}$$

also $|\widehat{f d\sigma}(\xi)| \leq \int_{S^{n-1}} |f(x)| d\sigma(x) \leq 1$

$$\Rightarrow |\widehat{f d\sigma}(\xi)| \lesssim (1+|\xi|)^{-1}$$

Note

$$|C_f| \leq C \sum_{|\alpha| \leq 2} \|D_\alpha f\|_\infty \quad (\text{as one needs to estimate the error term } O(x^{-\frac{n+1}{2}}))$$

Ex

Let $f_k(x) = e^{2\pi i k \cdot x}$, then at $\xi_k = k$

$$\widehat{f_k d\sigma}(\xi_k) = \int_{S^{n-1}} e^{-2\pi i x \cdot (\xi_k - k)} d\sigma(x) = \int_{S^{n-1}} d\sigma(x) = \sigma(S^{n-1}) \approx 1$$

Let $k_j \nearrow \infty$ rapidly, p.e. $k_{j+1} \geq C \cdot k_j^{1/2}$ and consider

$$f(x) = \sum_{j \geq 1} k_j^{-2} f_{k_j}(x) = \sum_{j \geq 1} k_j^{-2} e^{2\pi i k_j \cdot x}$$

Let $\xi_n = k_m$, then

$$\widehat{f d\sigma}(\xi) = \sum_{j < m} k_j^{-2} \widehat{f_{k_j}}(\xi) + k_m^{-1} \widehat{f_{k_m}}(\xi) + \sum_{j > m} k_j^{-2} \widehat{f_{k_j}}(\xi)$$

If $j < m$, then $|\widehat{f_{k_j}}(\xi)| \leq C_j k_m^{-\frac{n-1}{2}} \leq C b_j^2 k_m^{-\frac{n-1}{2}} \leq C b_j^2 k_m^{-\frac{1}{2}}$

Now $C \sum_{j < m} b_j^2 k_m^{-\frac{1}{2}} \leq C m k_m^{-\frac{1}{2}}$

If $j > m$ $|\widehat{f_{k_j}}(\xi)| \leq C$, thus $\sum_{j > m} b_j^{-2} |\widehat{f_{k_j}}(\xi)| \leq C k_m^{-1}$

$\Rightarrow |\widehat{f d\sigma}(\xi)| \approx (1 + O(m k_m^{-\frac{1}{2}})) \geq C > 0$

for $\xi = k_m$ for all m .

$\Rightarrow |\widehat{f d\sigma}(\xi)| \leq C(1 + |\xi|)^{-\epsilon}$ does not hold for any $\epsilon > 0$.

Restriction Conjecture (Stein)

If $f \in L^\infty(S^{n-1})$, then

$\|\widehat{f d\sigma}\|_{L^q(\mathbb{R}^n)} \leq C_q \|f\|_{L^\infty(S)}$ for all $q > \frac{2n}{n-1}$

Note $\frac{2n}{n-1}$ is best possible, as if $f \equiv 1$, then for $q \leq \frac{2n}{n-1}$

$\|\widehat{f d\sigma}\|_{L^q}^q = \|\widehat{d\sigma}\|_{L^q}^q \geq C \int_{\mathbb{R}^n} (1 + |\xi|)^{-\frac{n-1}{2} \cdot q} d\xi$
 $\geq C \int_{\mathbb{R}^n} (1 + |\xi|)^{-n} d\xi \geq C_n \int_{r \geq 1} r^{-1} dr = +\infty$

Fourier restriction

Thm (Tomas-Stein) If $f \in L^2(S^{n-1})$, then

$$\| \widehat{f d\sigma} \|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^2(S^{n-1})}, \text{ for } q \geq \frac{2n+2}{n-1} \quad (1)$$

Note • $|\widehat{f d\sigma}(\xi)| = \left| \int e^{-2\pi i x \cdot \xi} f(x) d\sigma(x) \right| \leq \int |f(x)| d\sigma(x) =$

$$\Rightarrow \| \widehat{f d\sigma} \|_{L^\infty} \leq \|f\|_{L^1(S^{n-1})} \leq C \|f\|_{L^2(S^{n-1})}$$

Thus if $q_0 \leq q \leq \infty$ and (1) holds for q_0 then it also holds for q .

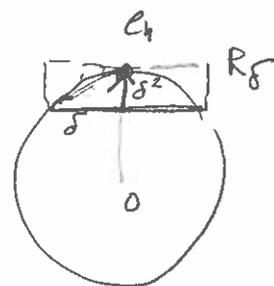
• $q \geq \frac{2n+2}{n-1}$ is best possible; Knapp example

Let $C_\delta = \{x \in S^{n-1}; 1 - x \cdot e_n \leq \delta^2\}$

As $|x - e_n|^2 = 2(1 - x \cdot e_n)$, we have

$$x \in C_\delta \Rightarrow |x - e_n|^2 \leq 2\delta^2 \Rightarrow |x - e_n| \leq \sqrt{2}\delta$$

$$\text{also } |x - e_n| \leq \delta \Rightarrow x \in C_\delta$$



Let $f = \mathbb{1}_{C_\delta}$ (indicator funct. of C_δ) $\Rightarrow \|f\|_{L^2(S^{n-1})} \leq C \delta^{\frac{n-1}{2}}$

Also $|\widehat{f d\sigma}(\xi)| = \left| \int_{C_\delta} e^{-2\pi i x \cdot \xi} d\sigma(x) \right| = \left| \int_{C_\delta} e^{-2\pi i (x - e_n) \cdot \xi} d\sigma \right|$

$$\geq \int_{C_\delta} \cos(2\pi (x - e_n) \cdot \xi) d\sigma(x)$$

Motivated by the uncertainty principle; since $\text{supp } f \subseteq R_\delta$

then \hat{f} behaves like a constant on R_δ^* which is a

$\delta^{-2} \times \delta^{-1} \times \dots \times \delta^{-1}$ rectangle; so $|\hat{f}| \geq C > 0$ on R_δ^* .

Indeed, if $|\zeta_n| \leq C^{-1} \delta^{-2}$ and $|\zeta_i| \leq C^{-1} \delta^{-1}$ then

$$|(x - e_n) \cdot \zeta| \leq |(x_{i-1}) \zeta_n| + \sum_{i=1}^{n-1} |x_i \zeta_i| \leq C^{-1} \delta^2 \delta^{-2} + (n-1) C^{-1} \delta \delta^{-1} \leq n C^{-1} \leq$$

thus $\cos(2\pi(x - e_n) \cdot \zeta) \geq \cos\left(\frac{\pi}{3}\right) \geq \frac{1}{2}$ for all $x \in R_\delta$.

This shows that $|\widehat{f d\sigma}(\zeta)| \geq \frac{1}{2} |C\delta| \geq C_n \delta^{n-1}$

$$\Rightarrow \|\widehat{f d\sigma}\|_{L^q(\mathbb{R}^n)} \geq C_n \delta^{n-1 - \frac{n+1}{q}} \quad (= |R_\delta^*|^{\frac{1}{q}} \delta^{n-1})$$

Thus if $\|\widehat{f d\sigma}\|_q \leq C \|f\|_2$ then

$$\delta^{n-1 - \frac{n+1}{q}} \leq C \delta^{\frac{n-1}{2}} \Rightarrow n-1 - \frac{n+1}{q} \geq \frac{n-1}{2}$$

$$\frac{n+1}{q} \leq \frac{n-1}{2} \Rightarrow q \geq \frac{2n+2}{n-1}$$

The TT* argument

Let (X, μ) and (Y, ν) be measure spaces.

Let $T: L^p(X, \mu) \rightarrow L^q(Y, \nu)$ be a linear operator, i.e.

$$\|Tf\|_{L^q(Y)} \leq C \|f\|_{L^p(X)}.$$

Define the inner product $\langle f, g \rangle_\mu = \int_X f(x) \overline{g(x)} d\mu$.

Then $\exists!$ linear operator $T^*: L^{q'}(Y, \nu) \rightarrow L^p(X, \mu)$ s.t.

$$\langle Tf, g \rangle_\mu = \langle f, T^*g \rangle_\nu \quad \forall f \in L^p(\mu), g \in L^{q'}(\nu)$$

Pf (Sketch)

Fix $g \in L^{q'}(\nu)$ and consider the linear functional

$$L_g(f) := \langle Tf, g \rangle_\mu$$

and note that $|L_g(f)| \leq \|Tf\|_{L^p(\mu)} \|g\|_{L^{q'}(\nu)} \leq C \|f\|_{L^p(\mu)}$ ($C = C(\|g\|_{L^{q'}(\nu)})$)

Thus $L_g \in L^p(\mu)^* = L^{p'}(\mu)$ (Riesz-repr. thm., follows from Radon-Nikodym)

i.e. $\exists! \beta \in L^{p'}(\mu)$ s.t. $L_g(f) = \langle Tf, \beta \rangle (= \langle Tf, g \rangle)$

Let $T^*: g \mapsto \beta$ and note that T^* is linear.

$$\text{Since } \|T^*g\|_{L^{p'}(\mu)} = \sup_{\|f\|_{L^p(\mu)} \leq 1} |\langle f, T^*g \rangle| = \sup_{\|f\|_{L^p(\mu)} \leq 1} |\langle Tf, g \rangle| \leq C \|Tf\|_{L^p(\mu)} \|g\|_{L^{q'}(\nu)} \leq C' \|g\|_{L^{q'}(\nu)}$$

Prop (i) $\|T\|_{L^p(\mu) \rightarrow L^q(\nu)} = \|T^*\|_{L^{q'}(\nu) \rightarrow L^{p'}(\mu)}$

(ii) If $p=2$, then $\|T\|_{L^2(\mu) \rightarrow L^q(\nu)}^2 = \|TT^*\|_{L^q(\nu) \rightarrow L^q(\nu)}$

df: By definition $(T^*)^* = T$, thus enough to show

$$\|T^*g\|_{L^{p'}(\mu)} \leq C \|g\|_{L^{q'}(\nu)} \text{ with } C := \|T\|_{L^p(\mu) \rightarrow L^q(\nu)}$$

(i) If $f \in L^p(\mu)$, $\|f\|_{L^p(\mu)} \geq 1$, then

$$|\langle f, T^*g \rangle_\mu| = |\langle Tf, g \rangle_\nu| \leq \|Tf\|_{L^q(\nu)} \|g\|_{L^{q'}(\nu)} \leq C \|g\|_{L^{q'}(\nu)}$$

$$\Rightarrow \|T^*g\|_{L^p(\mu)} \leq C$$

Now let for $f \in L^2(\sigma)$, $Tf(z) = \widehat{f d\sigma}(z)$
 Prop. $\|TT^*f\|_{L^q(\nu)} \leq \|T\|_{L^2 \rightarrow L^q} \|T^*f\|_{L^2} \leq \|T\|_{L^2 \rightarrow L^q} \|T^*\|_{L^{q'} \rightarrow L^2} \|f\|_{L^{q'}} = \|T\|_{L^2 \rightarrow L^q}^2 \|f\|_{L^{q'}}$

$$\|T^*f\|_{L^2(\mu)}^2 = \langle T^*f, T^*f \rangle_\mu = \langle TT^*f, f \rangle_\mu \leq \|TT^*f\|_{L^q(\mu)} \|f\|_{L^{q'}(\mu)} \leq \|TT^*\|_{L^{q'} \rightarrow L^q} \|f\|_{L^{q'}(\mu)}^2$$

$$\Rightarrow \|T^*\|_{L^{q'} \rightarrow L^2}^2 = \|T\|_{L^2 \rightarrow L^q}^2 \leq \|TT^*\|_{L^{q'} \rightarrow L^q}$$

Now, let $Tf = \widehat{f d\sigma}$. Then for $g \in S$ we have □

$$\langle Tf, g \rangle = \int_{\mathbb{R}^n} \widehat{f d\sigma}(z) \overline{g(z)} dz = \int_{\mathbb{R}^n} \int_{S^{n-1}} f(x) \overline{g(z)} e^{-2\pi i x \cdot z} dz d\sigma$$

$$= \int_{S^{n-1}} f(x) \int_{\mathbb{R}^n} \overline{g(z)} e^{+2\pi i x \cdot z} dz = \int_{S^{n-1}} f(x) \check{g}(x) d\sigma(x)$$

Thus $T^*g = \check{g}|_{S^{n-1}} = Rg = \langle f, \check{g}|_{S^{n-1}} \rangle_{d\sigma}$

Also $TT^*g(\gamma) = \int_{S^{n-1}} e^{-2\pi i x \cdot \gamma} T^*g(x) d\sigma(x) =$

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \gamma} e^{2\pi i x \cdot z} g(z) dz d\sigma(x) = \int_{\mathbb{R}^n} g(z) \widehat{d\sigma}(\gamma - z) dz$$

Thus $TT^*g = g * \widehat{d\sigma}$. Then it is enough to prove the following

Prop Let $q > \frac{2n+2}{n-1}$, then $\|g * \widehat{d\sigma}\|_{L^q(dx)} \leq C \|g\|_{L^{q'}(dx)}$

Idea: Since we have $|\widehat{d\sigma}(x)| \approx |x|^{-\frac{n-1}{2}}$ for $|x| \leq 1$
(and $\lesssim 1$ of $|x| \leq 1$)

decompose $\widehat{d\sigma}(x) = \varphi_0(x) \widehat{d\sigma}(x) + \sum_{j=1}^{\infty} \varphi_j(x) \widehat{d\sigma}(x)$

with $\text{supp } \varphi_0 \subseteq \{|x| \leq 2\}$, $\text{supp } \varphi_j \subseteq \{2^{j-1} \leq |x| \leq 2^{j+1}\}$

By interpolation it'll be easy to estimate

$$\| * \varphi_j \widehat{d\sigma} \|_{1 \rightarrow \infty} \leq \| \widehat{\varphi_j d\sigma} \|_{\infty} \leq C 2^{-j(n-1)} \quad (3)$$

and also

$$\| * \varphi_j \widehat{d\sigma} \|_{2 \rightarrow 2} \leq \| (\varphi_j \widehat{d\sigma})^\vee \|_{\infty} \leq C 2^j \quad (4)$$

and by interpolation (Riesz-Thorin thm) \Rightarrow

$$\| * \varphi_j \widehat{d\sigma} \|_{q' \rightarrow q} \leq C 2^{-j(n-1)\alpha} 2^{j(1-2\alpha)}$$

$$\frac{1}{q} = \frac{\alpha}{\infty} + \frac{1-\alpha}{2}$$

$$\text{if } \frac{1}{q} = \frac{1-\alpha}{2} = \alpha \frac{1}{\infty} + (1-\alpha)$$

which can be summed up in $j \geq 1$, if $q > \frac{2n+2}{n-1}$.

Proof

To prove (3)

$$\text{supp } \varphi_j \subseteq \{ 2^{j-1} \leq |z| \leq 2^{j+1} \} \Rightarrow |\varphi_j(z) \widehat{d\sigma}(z)| \lesssim 2^{-(j-1)\frac{n-1}{2}} \lesssim 2^{-j\frac{n-1}{2}}$$

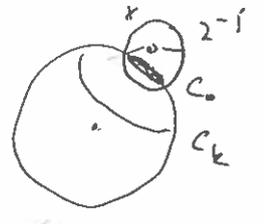
To prove (4)

$$\begin{aligned} (\varphi_j \widehat{d\sigma})^\vee(x) &= \int_{\mathbb{R}^n} e^{2\pi i x \cdot z} \varphi_j(z) \widehat{d\sigma}(z) = \int_{\mathbb{R}^n} \int_{S^{n-1}} e^{2\pi i (x-y) \cdot z} \varphi_j(z) d\sigma(y) dy \\ &= \int_{S^{n-1}} \widehat{\varphi}_j(y-x) d\sigma(y) = 2^{jn} \int_{S^{n-1}} \widehat{\varphi}(2^j(y-x)) d\sigma(y) = \\ &\leq C_N 2^{jn} \int_{S^{n-1}} (1 + 2^j|x-y|)^{-N} d\sigma(y) \quad (= \check{\varphi}_j * d\sigma(x)) \end{aligned}$$

Heuristics

$\widehat{\varphi}(x)$ is concentrated on $|x| \leq 1$
 $\Rightarrow \widehat{\varphi}(2^j|y-x|)$ is concentrated on $|x-y| \leq 2^{-j}$

$$\begin{aligned} \Rightarrow \widehat{\varphi}_j * d\sigma(x) &\lesssim 2^{jn} \sigma\{B_x(2^{-j}) \cap S^{n-1}\} \\ &\lesssim 2^{jn} 2^{-j(n-1)} = 2^j \end{aligned}$$



To prove this one needs a dyadic decomposition

of $x-y$; i.e. $C_k := \{y \in S^{n-1}; |x-y| \approx 2^{k-j}\}$
 $C_0 = \{y \in S^{n-1}; |x-y| \lesssim 2^{-j}\}$

Note $\sigma(C_0) \lesssim 2^{-j(n-1)}$, $\sigma(C_k) \lesssim 2^{(k-j)(n-1)}$

$$\begin{aligned} \check{\varphi}_j * d\sigma(x) &\leq C_N 2^{jn} \sum_{k=0}^{\infty} \int_{C_k} (1 + 2^j(2^{k-j}))^{-N} d\sigma(y) \\ &\leq C_N 2^{jn} \sum_{k=0}^{\infty} 2^{(k-j)(n-1)} 2^{-Nk} \\ &\leq C_N 2^j \sum_{k \geq 0} 2^{-k(N-n+1)} \leq C'_N 2^j \end{aligned}$$

Interpolation $TT^*f = f * \widehat{d\sigma} = \sum_{j \geq 0} S_j f$

$$\|TT^*\|_{q' \rightarrow q} \leq \sum_{j \geq 0} \|S_j\|_{q' \rightarrow q}$$

$$\|S_j\|_{1 \rightarrow \infty} \leq C 2^{-j \frac{(n-1)}{2}}$$

$$\|S_j\|_{q \rightarrow q} \leq ? \quad 1 + \frac{2}{n+1} = \frac{n+3}{n+1}$$

$$\|S_j\|_{2 \rightarrow 2} \leq C 2^j$$

Let $\frac{1}{q'} = \alpha + \frac{1-d}{2} = \frac{1+d}{2}$, then by Riesz-Thorin Thm

$$\|S_j\|_{q' \rightarrow q} \leq C 2^{-j \frac{(n-1)d}{2} + j(1-\alpha)} = C 2^{j[1 - \frac{n+1}{2}\alpha]}$$

Thus $\sum_{j \geq 0} \|S_j\|_{q' \rightarrow q} < \infty$ if $1 - \frac{n+1}{2}\alpha < 0 \Rightarrow \alpha > \frac{2}{n+1}$

$$\alpha > \frac{2}{n+1} \Leftrightarrow \frac{1+\alpha}{2} > \frac{n+3}{2(n+1)} \Leftrightarrow \frac{1}{q'} > \frac{n+3}{2n+2}$$

$$\Leftrightarrow \frac{1}{q} < \frac{n-1}{2n+2} \Leftrightarrow q > \frac{2n+2}{n-1}$$

□