

HA.

If  $I(\lambda) = \int_{\mathbb{R}^n} e^{-\pi i \lambda \phi(x)} a(x) dx$ , with  $\text{supp } a \subseteq U \ni p$ ,  $U$  suff.-small  
 $\nabla \phi(p) \neq 0$ ,  $\exists H_\phi(p)^{-1}$ , then

$$I(\lambda) = c_\phi \lambda^{-\frac{n}{2}} \left( a(p) + \sum_{j=1}^N a_j \lambda^{-j} + O(\lambda^{-N-1}) \right),$$

where  $a_j = \sum_{|\beta| \leq j} c_{\beta, \phi} D^\beta a(p)$ .

COR.

(i) If  $D^\beta a(p) = 0$  for all  $|\beta| \leq k$ , then  $|I(\lambda)| \leq \lambda^{-\frac{n+k}{2}}$

(ii) If  $\nabla \phi(p) = 0$  and  $H_\phi(p)$  invertible, then  $\forall k \geq 0$

$$D_\lambda^k (e^{\pi i \lambda \phi(p)} I(\lambda)) \leq C_k \lambda^{-(\frac{n}{2}+k)}, \text{ with } D_\lambda = \frac{d}{d\lambda}$$

Pf

(i) If  $j \leq \frac{k}{2}$  then  $a_j = \sum_{|\beta| \leq j} c_{\beta, \phi} D^\beta a(p) = 0 \Rightarrow |I(\lambda)| \leq \lambda^{-\frac{n}{2}} \lambda^{-j}$

(ii)  $D_\lambda^k \int e^{-\pi i \lambda (\phi(x) - \phi(p))} a(x) dx = (-\pi i)^k \int e^{-\pi i \lambda (\phi(x) - \phi(p))} (\phi(x) - \phi(p))^k b(x) dx$   
 $= -(-\pi i)^k e^{\pi i \lambda \phi(p)} \int e^{-\pi i \lambda \phi(x)} b(x) dx$ , with

$$b(x) = (\phi(x) - \phi(p))^k a(x),$$

Let  $|\alpha| \leq 2k$ , then by the product rule

$D^\alpha (\phi(x) - \phi(p))^k = D^\alpha \left[ \prod_{l=1}^k (\phi(x) - \phi(p)) \right]$  is a linear comb.  
of terms  $\prod_{i=1}^k D_{x_i}^{\beta_i} (\phi(x) - \phi(p))$  where  $\sum_{i=1}^k |\beta_i| = |\alpha| \leq 2k$

$$\Rightarrow \exists i: |\beta_i| \leq 1 \Rightarrow D^{\beta_i} (\phi(x) - \phi(p))|_{x=p} = 0 \Rightarrow D^\alpha b(p) = 0$$

$\# |\alpha| < k \quad \square$

## Fourier transform of surface carried measures

• The sphere Let  $S^{n-1} = \{x \in \mathbb{R}^n; |x|=1\} \subseteq \mathbb{R}^n$ . Then  $S$  is a hypersurface, i.e. a smooth  $n-1$ -dim submanifold of  $\mathbb{R}^n$ .

This means that  $\forall p \in S^{n-1} \exists P \in V \subseteq S^{n-1}$  open and  $U \subseteq \mathbb{R}^{n-1}$  open with a 1-1 onto map  $F: U \rightarrow V$  (called a local coord chart)

Ex Let for  $P = \pm e_n; e_n = (0, \dots, 0, 1)$

$$U^\pm = D(0, 1 - \frac{1}{n}), F_n^\pm(x) = (x, \pm \sqrt{1 - |x|^2}); U_n \rightarrow V_n^\pm \subseteq S^{n-1}$$

Similarly for any  $1 \leq j \leq n; e_j = (0, \dots, 0, \overset{j}{1}, 0, \dots)$

$$F_j^\pm(x) = (x_1 - x_{j+1} \pm \sqrt{1 - |x|^2}, x_{j+1}, \dots, x_n), \text{ writing } D(0, 1 - \frac{1}{n}) =$$

HW: Show that  $S^{n-1} = \bigcup_{j=1}^n V_j^\pm$ ; and  $\pm e_n \in V_n^\pm$  but  $\pm e_n \notin V_j^\pm \quad \forall j \neq n$ .

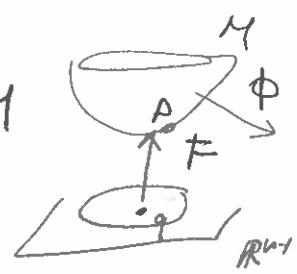
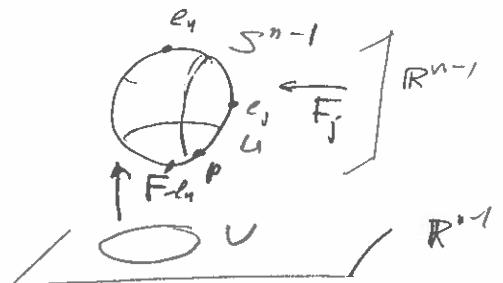
Claim: Let  $\Phi \in C^\infty(\mathbb{R}^n)$ ,  $M \subseteq \mathbb{R}^n$  smooth hypersurface.

Let  $p \in M$ ,  $F: U \rightarrow V \subseteq M$ ,  $p \in V$  be a coord-chart.

Then  $\nabla_q (\Phi \circ F) = 0 \Leftrightarrow$  with  $F(q) = p$ .

i.e.  $\nabla \Phi(p)$  is perpendicular to  $T_p M$

tangent space  $T_p M$ .



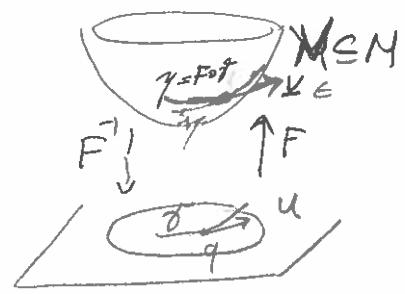
H.A.

14/

Pf Let  $\gamma(t) \subseteq U$ ,  $\gamma(0)=q$  be a (smooth) curve.

Then  $\gamma(t) = F\gamma(t) \subseteq V$ ,  $\gamma(0)=p \Rightarrow$

$\gamma'(0) \in T_p M$  (and  $T_p M$  generated by)



Now  $D_q(\phi \circ F) = 0 \Leftrightarrow \gamma'(0)$

$$\frac{d}{dt} (\phi \circ F)(\gamma(t))|_{t=0} = D_q(\phi \circ F) \cdot \gamma'(0) = 0$$

Also

$$\frac{d}{dt} (\phi \circ F \circ \gamma)|_{t=0} = \frac{d}{dt}|_{t=0} (\phi \circ \gamma)(t) = D\phi_p \cdot \gamma'(0) \quad \forall \gamma(t)$$

and

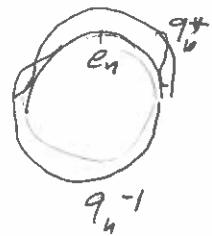
$$D\phi_p \cdot \gamma'(0) = 0 \quad \forall \gamma(t) \subseteq M, \gamma(0)=p \Leftrightarrow D_p \phi \perp T_p M.$$

Now let  $\sigma$  be the surface area measure on  $S^{n-1}$ .

Note  $\hat{\sigma}(3) = \hat{\sigma}(u_3)$  for all  $U$  rotations (i.e.  $U^T U = I$ );

$$\begin{aligned} \text{as } \hat{\sigma}(u_3) &= \int_{S^{n-1}} e^{-2\pi i x \cdot u_3} d\sigma(x) = \int_{S^{n-1}} e^{-2\pi i u^T x \cdot 3} d\sigma(x) \\ &= \int_{S^{n-1}} e^{-2\pi i y \cdot 3} d\sigma(y) = \hat{\sigma}(3), \end{aligned}$$

Thus if  $|3| = \lambda$  then  $\hat{\sigma}(3) = \hat{\sigma}(\lambda \cdot e_n)$ .



Partition of unity

There exists functions

$$\text{such that } 0 \leq q_k^\pm \leq 1, \sum_k q_k^+ + q_k^- = 1 \text{ on } S^{n-1}, \quad q_k^\pm \in C^\infty(\mathbb{R}^n)$$

$$\text{but } q_k^+(e_n) = 0 \quad \forall k < n.$$

$$q_k^\pm(x) = 0 \text{ if } x \notin U_k^\pm,$$

For simplicity let's relabel  $q_1^\pm, \dots, q_{2n}^\pm$  as  $V_1^\pm, \dots, V_{2n}^\pm$  s.t.  $q_{2n}(e_n) = 1$

Then  $\int_{S^{n-1}} \phi(x) d\sigma(x) = \sum_{k=1}^{2n} \int_{S^{n-1}} q_k(x) \phi(x) d\sigma(x) = \sum_{k=1}^{2n} \int_{V_k} \phi(x) \underbrace{q_k(x) d\sigma(x)}_{x = F_k(y), y \in U} dy$  for  $k < 2n$

$$= \sum_{k=1}^{2n} \int_{U_k} (\phi \circ F_k)(y) b_k(y) dy = \sum_{k=1}^{2n} \int_{\mathbb{R}^{n-1}} (\phi \circ F_k)(y) b_k(x) dx.$$

Let  $\phi(x) = e^{i\lambda x \cdot e_n} = e^{i\lambda x_n}$ . Then

$$\int_{S^{n-1}} \phi(x) d\sigma(x) = \int_{S^{n-1}} e^{i\lambda x \cdot e_n} d\sigma(x) = \hat{\phi}(\lambda e_n),$$

But  $\partial_{x_j} \phi(x) = \bar{e}^{2\pi i \lambda x_n} \quad \partial_{x_j}(\lambda e_n) = 0 \text{ if } j \neq n$   
 $\partial_{x_n} \phi(x) = \bar{e}^{2\pi i \lambda x_n} \quad \left. \Rightarrow \right. \nabla_p \phi = \bar{e}^{2\pi i \lambda p_n} e_n$

Thus  $\nabla_p \phi \perp T_p S^{n-1} \Leftrightarrow p = e_n \text{ or } p = -e_n$

$$\Rightarrow \nabla_q (\phi \circ F_k) \neq 0 \quad \forall q \in U_k \text{ for } k < 2n$$

$$\Rightarrow \int_{\mathbb{R}^{n-1}} \bar{e}^{2\pi i \lambda x_n} q_k(x) d\sigma(x) \leq C_N |\lambda|^{-N} \quad \forall k < 2n$$

Thus to get an asymptotics, we only need to calculate

$$\int_{S^{n-1}} \bar{e}^{2\pi i \lambda (F_n^\pm(x) \cdot e_n)} b_n(x) dx = \int \bar{e}^{\frac{2\pi i \lambda}{\sqrt{1-|x|^2}}} b_n(x) dx + \int e^{-i\lambda \sqrt{1-|x|^2}} b_n(x) dx$$

This is an oscillatory integral with phase  $f(x) = 2(1-|x|^2)^{-\frac{1}{2}}$ .

Note

$$\nabla f(x) = \frac{2x}{(1-|x|^2)^{\frac{3}{2}}} = 0 \iff x = 0 \text{ or } |x| = \frac{1}{\sqrt{n}}$$

Using the coord. chart  $x = (y, \sqrt{1-y^2})$ ,  $d\sigma(x) = J(y) dy$  [15/1]

Note

$$J(0) = 1, \quad \psi_n(x) d\sigma(x) = b(y) dy = \psi_n(y, \sqrt{1-y^2}) J(y) dy$$

$$\psi_n(e_n) = 1 \Rightarrow \psi_n(0, (0, 1)) J(0) = 1 \Rightarrow b(0) = 1$$

$$I(\lambda) = \int e^{-2\pi i \lambda \sqrt{1-y^2}} b(y) dy; \quad \phi(y) = \sqrt{1-y^2}, \quad \phi(0) = 1$$

$$e^{i\pi((\lambda - \frac{n-1}{2}))}$$

$$\nabla \phi(y) = \frac{-y}{\sqrt{1-y^2}} \Rightarrow \nabla \phi(0) = 0$$

$$\text{Thus } H_{\phi}(0) = -2 \cdot I, \quad \sigma(H_{\phi}(0)) = -2(n-1)$$

$$I(\lambda) = e^{-2\pi i \lambda} e^{-\frac{\pi i (n-1)}{4}} \lambda^{-\frac{n-1}{2}} \cdot 1 + O(\lambda^{-\frac{n+1}{2}})$$

$$\Rightarrow \hat{\sigma}(\lambda e_n) = 2 \lambda^{-\frac{n-1}{2}} \cos(2\pi(\lambda - \frac{n-1}{2})) + O(\lambda^{-\frac{n+1}{2}})$$

$$\Rightarrow \hat{\sigma}(\bar{z}) = 2 |\bar{z}|^{-\frac{n-1}{2}} \cos(2\pi(|\bar{z}| - \frac{n-1}{2})) + O(|\bar{z}|^{-\frac{n+1}{2}})$$

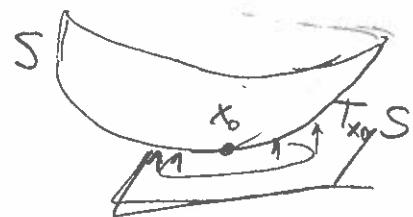
Hypersurfaces with non-zero Gaussian curvature

Let  $S \subseteq \mathbb{R}^n$ ,  $\dim S = n-1$ ,  $x_0 \in S$

By a translation move  $x_0 \rightarrow 0$ , and

by a rotation  $T_{x_0}S = T_0S$  to  $x_n = 0$ ;

then one can write :  $S = \{(x_1, \dots, x_{n-1}, \phi(x_n, x_{n-1}))\}$



S has non-zero Gaussian curvature if  $H_{\phi}(0)$  is invertible

Note

• Since  $T_0 S = \{x_n=0\}$ , it follows  $\nabla \phi(0) = 0$ .

Indeed :  $\forall v = (v_1, \dots, v_{n-1})$  we have  $\gamma(t) = (tv, \phi(tv)) \in S$

$$\Rightarrow \gamma'(0) = (v, \nabla \phi(0) \cdot v) \in T_0 S \Rightarrow \nabla \phi(0) \cdot v = 0 \quad \forall v \\ \Rightarrow \nabla \phi(0) = 0.$$

- Consider the map  $x \mapsto \nabla \phi(x) = (\partial_{x_1} \phi(x), \dots, \partial_{x_{n-1}} \phi(x))$ ;  $0 \mapsto 0$   
the Jacobian of this map :  $J_{\nabla \phi}(x) = H_{\phi}(x)$   
 $\Rightarrow J_{\nabla \phi}(0) = H_{\phi}(0)$  is invertible  
 $\Rightarrow \nabla \phi: \underset{\psi}{U} \rightarrow \underset{\psi}{V}$  diffeomorphism.  
 $\Rightarrow \forall \bar{y} \in V \exists! x = x(\bar{y}) \text{ s.t. } \nabla \phi(x) = \bar{y}.$

Thm Let  $S \subseteq \mathbb{R}^n$  be a smooth hypersurface, let  
 $\psi \in C_0^\infty(S)$ ,

If  $\forall x \in \text{supp } \psi$ ,  $S$  has non-vanishing Gaussian curvature at  $x$ , then

$$\widehat{\psi d\sigma_S}(3) = \int_S e^{-2\pi i x \cdot \vec{z}} \psi(x) d\sigma_S(x) = O(|3|^{-\frac{n-1}{2}}).$$

Pf Assume  $x_0 \in S$  and  $\text{supp } \psi \subseteq V_{x_0}$ ,  $U$  suff small

- By a translation and rotation assume

$$S = \{x, \phi(x)\}; \text{ with } \phi(0) = \nabla \phi(0) = 0$$

Indeed: •  $x \rightarrow x - x_0$  then  $\widehat{(\psi d\sigma)}_{S-x_0}(z) = \widehat{\psi d\sigma}(z) e^{2\pi i x_0 \cdot z}$   
 $S \mapsto S - x_0$

$$\bullet S \mapsto S \circ U \text{ then } \widehat{(\psi d\sigma) \circ U}(z) = \widehat{\psi d\sigma}(Uz)$$

- Let  $z = \lambda \cdot \gamma$  with  $|\lambda| = |\gamma|$  and  $\gamma \in S^{n-1}$

$$\text{Then } \widehat{\psi d\sigma}(\lambda \cdot \gamma) = \int e^{-2\pi i \lambda (x_1 \gamma_1 + \dots + x_{n-1} \gamma_{n-1} + \phi(x) \gamma_n)} a(x) dx$$

$$\text{as } \langle (x_1, \dots, x_{n-1}, \phi(x)), (\gamma_1, \dots, \gamma_n) \rangle = \sum_{j=1}^{n-1} x_j \gamma_j + \phi(x) \gamma_n =: \Phi(x, \gamma)$$

Also  $d\sigma_S(x) = (1 + |\nabla \phi(x)|^2)^{\frac{1}{2}} dx \leftarrow \text{though this does not matter}\right.$   
just the fact  $\psi(x) d\sigma_S(x) = a(x) dx$  with  $a \in C_c^\infty$

$\text{Supp } a \subseteq U$ ,  $0 \in U$ ,  $U$  "suff small"; meaning that

$\nabla \phi$  is a diffeom.  $\nabla \phi: U \rightarrow V$

- Now  $\nabla_x \Phi(x, \gamma) = (\partial_{x_1} \Phi(x, \gamma), \dots, \partial_{x_{n-1}} \Phi(x, \gamma)) =$

$$= (\gamma_1 + \partial_{x_1} \phi(x) \cdot \gamma_n, \dots, \gamma_{n-1} + \partial_{x_{n-1}} \phi(x) \cdot \gamma_n) = 0$$

if and only if  $\nabla \phi(x) = -\frac{\gamma}{\gamma_n}$ ;  $\gamma' = (\gamma_1, \dots, \gamma_{n-1})$

Case 1:  $-\frac{\gamma'}{\gamma_n} \in V$  so  $-\frac{\gamma'}{\gamma_n}$  is in a small ngh. of 0

Note this means that  $\gamma' \approx 0$  and  $\gamma_n \approx \pm 1$  so  
 $\gamma \approx (0, -1, 0, 1)$  or  $\gamma \approx (0, -1, 0, -1)$

Then  $\exists! x = x(\gamma) \in U$  s.t.  $\nabla_x \Phi(x(\gamma), \gamma) = 0$ .

$$\Rightarrow \widehat{\psi d\sigma}(\lambda \gamma) = \int e^{-2\pi i \lambda} \Phi(x, \gamma) a(x) dx = O(\lambda^{-\frac{n-1}{2}})$$

(in fact  $\approx c \lambda^{-\frac{n-1}{2}} + O(\lambda^{-\frac{n+1}{2}})$ )

Case 2:  $-\frac{\gamma'}{\gamma_n} \notin V$  then  $\nabla_x \Phi(x, \gamma) \neq 0 \quad \forall x \in \text{supp } a$

$$\Rightarrow |\widehat{\psi d\sigma}(\lambda \gamma)| \leq C_N \lambda^{-N} \quad \forall N$$

□

Restriction problem

Let  $f: S^{n-1} \rightarrow \mathbb{C}$ , and consider

$$\widehat{f d\sigma}(z) = \int_{S^{n-1}} f(x) e^{-2\pi i x \cdot z} d\sigma(x)$$

If  $f \in C^\infty(S^{n-1})$ , then we have seen that

$$|\widehat{f d\sigma}(z)| \leq C_f |z|^{-\frac{n-1}{2}}$$

also  $|\widehat{f d\sigma}(z)| \leq \int_{S^{n-1}} |f(x)| d\sigma(x) \leq 1$

$$\Rightarrow |\widehat{f d\sigma}(z)| \leq (1+|z|)^{-\frac{1}{2}}$$

Note  $|C_f| \leq C \sum_{|\alpha| \leq 2} \|D_\alpha f\|_\infty$  (as one needs to estimate the error term  $O(|z|^{-\frac{n+1}{2}})$ )

Ex Let  $f_k(x) = e^{2\pi i k \cdot x}$ , then at  $z_k = k$

$$\widehat{f_k d\sigma}(z_k) = \int_{S^{n-1}} e^{-2\pi i x \cdot (z_k - k)} d\sigma(x) = \int_{S^{n-1}} d\sigma(x) = \sigma(S^{n-1}) \approx 1$$

Let  $k_j \nearrow \infty$  rapidly, p.e.  $k_{j+1} > C \cdot k_j^{10}$  and consider

$$f(x) = \sum_{j \geq 1} k_j^{-2} f_{k_j}(x) = \sum_{j \geq 1} k_j^{-2} e^{2\pi i k_j \cdot x}$$

Let  $z_n = k_m$ , then

$$\widehat{f d\sigma}(z) = \sum_{j \leq m} k_j^{-2} \widehat{f_{k_j}}(z) + k_m^{-1} \widehat{f_{k_m}}(z) + \sum_{j > m} k_j^{-1} \widehat{f_{k_j}}(z)$$

If  $j < m$ , then  $|\widehat{f_{k_j}}(\vec{z})| \leq c_j k_m^{-\frac{n-1}{2}} \leq c b_j^2 k_m^{-\frac{n-1}{2}} \leq C b_j^2 k_m^{-\frac{1}{2}}$

Now  $C \sum_{j < m} b_j^2 k_j^2 k_m^{-\frac{1}{2}} \leq C k_m^{-\frac{1}{2}}$

15/6

If  $j > m$   $|\widehat{f_{k_j}}(\vec{z})| \leq C$ , thus  $\sum_{j > m} b_j^{-2} |\widehat{f_{k_j}}(\vec{z})| \leq C k_m^{-\frac{2}{2}}$

$\Rightarrow |\widehat{f d\sigma}(\vec{z})| \approx (1 + O(m k_m^{-\frac{1}{2}})) \geq C > 0$

for  $\vec{z} = k_m$  for all  $m$ .

$\Rightarrow |\widehat{f d\sigma}(\vec{z})| \leq C(1 + |\vec{z}|)^{-\varepsilon}$  does not hold for any  $\varepsilon > 0$ .

### Restriction Conjecture (Stein)

$$\|\widehat{f d\sigma}\|_{L^q(\mathbb{R}^n)}^q \leq C_q \|f\|_{L^\infty(S)}^q \quad \text{for all } q > \frac{2n}{n-1}$$

Note  $\frac{2n}{n-1}$  is best possible, as if  $f \equiv 1$ , then for  $q \leq \frac{2n}{n-1}$

$$\|\widehat{f d\sigma}\|_{L^q}^q = \|\widehat{d\sigma}\|_{L^q}^q \geq C \int_{\mathbb{R}^n} (1 + |\vec{z}|)^{-\frac{n-1}{2} \cdot q} d\vec{z}$$

$$\geq C \int_{\mathbb{R}^n} (1 + |\vec{z}|)^{-n} d\vec{z} \geq C_n \int_{r \geq 1} r^{-1} dr = +\infty$$

Thm (Tomas-Stein) If  $f \in L^2(S^{n-1})$ , then

$$\|\widehat{f d\sigma}\|_q \leq C \|f\|_{L^2(S^{n-1})}, \text{ for } q \geq \frac{2n+2}{n-1}$$

and the range of  $q$  is best possible.