

H.A.: Oscillatory integrals

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Thm (Implicit function theorem). Let $\Omega \subseteq \mathbb{R}^n$ open, $p \in \Omega$.

Let $f: \Omega \rightarrow \mathbb{R}$ be a C^∞ function, such that $\nabla f(p) \neq 0$.

Then $\exists \delta \in U$ and $p \in V$ and a diffeom $\Phi: U \rightarrow V$, $\Phi(p) = p$

s.t. $f \circ \Phi(x) = f(p) + x_n$

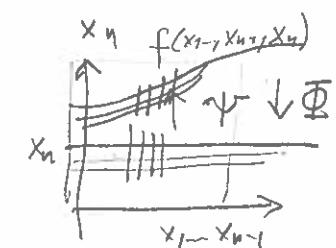
Pf (Sketch) Can assume, that $\partial_{x_n} f(p) \neq 0$. Let

$$\bar{\Psi}(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x)) \text{, then } J_p \bar{\Psi} \neq 0$$

$$\Phi := \bar{\Psi}^{-1}; \text{ let } (y_1, \dots, y_n) = \bar{\Psi}(x_1, \dots, x_n)$$

$$\text{then } x_1 = y_1, \dots, x_{n-1} = y_{n-1}, x_n = f(y_1, \dots, y_{n-1})$$

$$\Rightarrow f \circ \bar{\Psi}(x_1, \dots, x_n) = f(y_1, \dots, y_{n-1}) = x_n \text{, wlog } f(p) = 0.$$



$$(f(x) \mapsto g(x) = f(x) - p)$$

Thm (Morse Lemma) Let $\Omega \subseteq \mathbb{R}^n$ open, $f: \Omega \rightarrow \mathbb{R}$ is C^∞ ,

$$p \in \Omega, \nabla f(p) = 0 \text{ and } H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right) \text{ is invertible.}$$

Then $\exists \delta \in U, p \in V$ and $\Phi: U \rightarrow V$ diffeom, $\Phi(p) = p$

s.t. $f \circ \Phi(x) = f(p) + \sum_{j=1}^k x_j^2 - \sum_{j=k+1}^n x_j^2$

Morse lemma (sketch)

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$$(n=2) \quad \text{WLOG } f(0) = 0 \quad (f \mapsto f \circ \varphi, \varphi(0) = p)$$

$$f(x) = x_1 g_1(x) + x_2 g_2(x)$$

$$\Rightarrow \partial_{x_i} f(0) = g_i(0) = 0 \quad \text{from } \nabla f(0) = 0$$

$$g_1(x) = x_1 h_{11}(x) + x_2 h_{12}(x), \quad g_2(x) = x_1 h_{21}(x) + x_2 h_{22}(x)$$

$$f(x) = x_1^2 h_{11}(x) + 2x_1 x_2 h_{12}(x) + x_2^2 h_{22}(x)$$

$$\Rightarrow H_f(0,0) = \begin{pmatrix} h_{11}(0) & h_{12}(0) \\ h_{21}(0) & h_{22}(0) \end{pmatrix}$$

i) Assume

$$H_f(0,0) > 0 \quad (n=2) \quad \partial_{x_1}^2 f(0,0) > 0$$

$$\Rightarrow h_{11}(x) > 0, \quad \det(h_{ij}(x)) > 0 \quad \text{for } x \in U, \exists 0 \in U \text{ neighborhood}$$

$$\Rightarrow f(x) = \psi_1(x)^2 + \psi_2(x)^2 \quad ; \quad \psi_1 = x_1 h_{11}^{\frac{1}{2}} + x_2 \frac{h_{12}}{h_{11}^{\frac{1}{2}}} \quad ; \quad \psi_2 = x_2 \left(h_{22} - \frac{h_{12}}{h_{11}^{\frac{1}{2}}} \right)$$

The map $\psi = (\psi_1, \psi_2)$ has

$$\Rightarrow \text{let } \Phi = \psi^{-1}, \text{ then } J\psi(0) \neq 0 \Rightarrow \psi_2 = x_2 \frac{\det(\Phi)}{h_{11}^{\frac{1}{2}}} > 0.$$

$$f \circ \Phi(\psi_1, \psi_2) = f(x_1, x_2) = \psi_1(x)^2 + \psi_2(x)^2$$

$$= y_1^2 + y_2^2$$

□

Note p is called a non-deg. crit. point.

If p is a degenerate critical point and $n > 1$ then the behavior of $I(\lambda)$ as $\lambda \rightarrow \infty$ is quite complicated and depends on the so-called Newton polyhedron of ϕ .

Prop 1 If $f: S \rightarrow \mathbb{R}$, $p \in S$, $\nabla f(p) \neq 0$ and $p \in \text{supp } a$ is suff. small, then $\forall N \in \mathbb{N}$

Pf: Let $\Phi: U \rightarrow V$ s.t. $f \circ \Phi = x_n + c$, $c = f(p)$.
Assuming $\text{supp } a \subseteq V$, Subst.: $x = \Phi(y)$

$$I(\lambda) = \int_V e^{-\pi i \lambda} f(x) a(x) dx = \int_U e^{-\pi i \lambda} f \circ \Phi(y) J_\Phi(y) (a \circ \Phi)(y) dy$$

Let $x = \Phi(y)$, $dx = J_\Phi(y) dy$

$$= \int_{\mathbb{R}^n} e^{-\pi i \lambda (y_n + c)} b(y) dy = e^{-\pi i \lambda c} \int_{\mathbb{R}^{n-1}} \widehat{b}(y_1, \dots, y_{n-1}, \frac{1}{2}) dy_1 dy_{n-1}$$

Indeed $(2\pi i \lambda)^N \widehat{b}(y_1, \dots, y_{n-1}, \lambda) \leq C_N \lambda^{-N}$ as $\lambda \rightarrow \infty$.

$$\text{thus } (2\pi \lambda)^N \int_{\mathbb{R}^{n-1}} |\widehat{b}(y_1, \dots, y_{n-1}, \lambda)| dy_1 dy_{n-1} \leq \int_{\mathbb{R}^n} |\partial_{x_n}^N b(y_1, \dots, y_n)| \leq C_N$$

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Prop 3 Let T $n \times n$ real symm, and assume $\det T \neq 0$,
a.e. case.

Then for $I(\lambda) := \int e^{-\pi i \lambda \langle Tx, x \rangle} a(x) dx$, we have

$$I(\lambda) = e^{-\frac{\pi i \sigma}{4}} |\det T|^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} \left(a(0) + \sum_{j=1}^N \lambda^{-j} D_j a(0) + O(\lambda^{-N-1}) \right)$$

where $\sigma = \operatorname{sgn} T$, D_j is a diff. op. of order $2j$.

Pf: Let $g_T(x) = e^{-\pi i \lambda \langle Tx, x \rangle}$ Gaussian. We have proved,
 that $\widehat{g}_T(z) = e^{-\frac{\pi i \sigma}{4}} |\det T|^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} e^{\pi i \frac{\langle T^{-1}z, z \rangle}{\lambda}}$

Again, write

$$e^{\pi i \lambda^{-1} \langle T^{-1}z, z \rangle} = 1 + \sum_{j=1}^N \frac{(\pi i)^j \lambda^{-j} \langle Tz, z \rangle^j}{j!} + O\left(\frac{|z|^{2N+1}}{\lambda^{N+1}}\right)$$

And note that

$$\int \frac{\langle \pi i Tz, z \rangle^j}{j!} \widehat{a}(z) dz = \int \widehat{D_j a}(z) dz = \widehat{D_j a}(0),$$

for some diff. op. of order $2j$.

Note

Suppose $\phi \in C^\infty$, $\nabla \phi(p) = 0$ and $G: U \rightarrow V$, $G(p) = p$

Then

$$H_{\phi \circ G}(0) = G_*(0)^T H_\phi(p) G_*(0), \text{ where}$$

$$G_*(0) = \left\{ \frac{\partial G_i}{\partial x_j}(0) \right\}_{i,j=1}^n \quad \text{writing } G = (G_1, G_n)$$

Pf:

By chain rule:

$$\text{Let } \gamma: \mathbb{R} \rightarrow \mathbb{R}^n, \gamma(0) = p \quad \left(\frac{d}{dt} \right)^2 \Big|_{t=0} \phi(\gamma(t)) = \langle H_\phi(p) \gamma'(0), \gamma'(0) \rangle$$

Now consider $\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$, $\gamma(0) = 0$, $G(\gamma(t)) = p \Rightarrow \gamma(t) = p$

$$\left(\frac{d}{dt} \right)^2 \Big|_{t=0} (\phi \circ G)(\gamma(t)) = \left(\frac{d}{dt} \right)_0^2 (\phi \circ \gamma(t)) =$$

$$= \langle H_\phi(p) \gamma'(0), \gamma'(0) \rangle = \langle H_\phi(p) G_*(0) \gamma'(0), G_*(0) \gamma'(0) \rangle$$

$$= \langle G_*(0)^T H_\phi(p) G_*(0) \gamma'(0), \gamma'(0) \rangle = \langle H_{\phi \circ G}(0) \gamma'(0), \gamma'(0) \rangle$$

Prop

Let $\phi \in C^\infty$ s.t. $\nabla \phi(p) = 0$, $H_\phi(p)$ invertible, $a \in C^\infty$

$$\text{Let } \Delta = 2^{-n} / |\det H_\phi(p)|, \sigma = \text{sgn } H_\phi(p).$$

Then for $I(\lambda) := \int e^{-\pi i \lambda \phi(x)} a(x) dx$,

we have

$$I(\lambda) = e^{-\pi i \lambda \phi(p)} e^{-\frac{\pi i \sigma}{n} \Delta^{-\frac{1}{2}} \lambda^{-\frac{n}{2}}} \left(a(p) + \sum_{j=1}^N \tilde{J}^j D_j a(p) + O(\lambda^{-n-1}) \right)$$

where D_j is differential operator of order $\leq 2j$, with coefficients depending on ϕ .

Pf wlog assume $\phi(p) = 0$, by $\phi \rightarrow \phi - \phi(p)$.

Let G be a diffeom, s.t. $\partial_{x_i} G = \langle T_{x_i}, v \rangle$ for all i .

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad H_{\langle T_{x_i}, v \rangle}(0) = (-1)^{\text{sgn } T} 2^n \leftarrow \partial_{x_i}^2(x_i^2) = 2$$

then $|J_G(0)|^2 \cdot \det(H_{\phi(p)}) = 2^n (-1)^{\text{sgn } T} \lambda_{\phi(p)}^{-2} \Rightarrow |J_G(0)| = \Delta^{-\frac{1}{2}}$

Then $I(\lambda) = \int e^{-\pi i \lambda \langle \phi \circ G(y), y \rangle} \underbrace{\frac{a \circ G(y)}{b(y)} J_G(y)}_{b(y)} dy$

$$= \int e^{-\pi i \lambda \langle Ty, y \rangle} b(y) dy = e^{-\frac{\pi i \lambda}{4}} \lambda^{-\frac{n}{2}} \left(b(0) + \sum_{j \in N} \lambda^{-j} D_j b(0) + O(\lambda^{-n}) \right)$$

Also, $b(0) = a(G(0)) J_G(0) = a(p) \Delta^{-\frac{1}{2}}$, moreover

$D_j b(0) = D_j((a \circ G), J_G)(0)$ is a lin. comb. of partial derivatives of a at p mult. by functions depending on G and hence on ϕ

Note Most important is $|I(\lambda)| \lesssim \lambda^{-\frac{n}{2}}$ □

and

$$F(\lambda) = e^{-\frac{\pi i \lambda}{4}} \Delta^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} a(p) + O(\lambda^{-\frac{n}{2}-1})$$