Chapter 5 The Trigonometric Functions
Section 5.1 Angles

Initial side
Terminal side
Standard position of an angle

Positive angle
Negative angle

Coterminal Angles

Acute angle
Obtuse angle
Complementary angles
Supplementary angles

Definition of Radian Measure
One radian is the measure of the central angle of a circle subtended by an arc equal in length to the radius of the circle.

\[180^\circ = \pi \text{ radians}\]

Formula for the Length of a Circular Arc: \[s = r \cdot \theta\] where \(\theta\) must be in radian measure.

Formula for the Area of a Circular Sector: \[A = \frac{1}{2} r^2 \cdot \theta\] where \(\theta\) must be in radian measure.
Example 1: An arc of length 15 cm on a circle subtends a central angle of 21 degrees. Determine the area $A$, in square centimeters, of the sector outlined in green. Your answer must be correct to three decimal places.

Example 2: The figure shows a circular sector with radius $r$ and central angle $T$, in radians. If the total perimeter of the sector is 8, express $r$ as a function of $T$.

Example 3: Assume that the Earth is a sphere of radius 4000 miles, and longitude lines are circles with center located at the center of the Earth. If the latitude reading of Athens Georgia is 34.0 ° N, how far north of the equator is Athens? (Enter an exact expression or one correct to 3 decimal places.)

Example 4:
The figure shows a sector with radius $r$ and angle $\theta$ in radians. The total perimeter of the sector is 95 cm.
(a) Express $\theta$ as a function of $r$.
$\theta = $

(b) Express the area of the sector as a function of $r$.
area =

(c) For what radius $r$ is the area a maximum? (As usual, decimal approximations will be marked incorrect)
$r = $

(d) What is the maximum area? (Give a symbolic answer; decimal approximations will be marked incorrect)
area =
Section 5.2 Trigonometric Functions of Angles

1. Definition of Trigonometric Functions of an Acute Angle

\[ \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H} \]
\[ \csc \theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{H}{O} \]
\[ \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H} \]
\[ \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{H}{A} \]
\[ \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{O}{A} \]
\[ \cot \theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{A}{O} \]

2. Special Values

Theorem: In a 30-60 degree right triangle, the length of the side opposite the 30 degree angle is the half the length of the hypotenuse.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{\sqrt{2}}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
</tr>
</tbody>
</table>

Proof:
3. Identities

Set 1: Pythagorean Identities
\[
\sin^2 \theta + \cos^2 \theta = 1
\]
\[
\tan^2 \theta + 1 = \sec^2 \theta
\]
\[
\cot^2 \theta + 1 = \csc^2 \theta
\]

Set 2:
\[
\frac{\sin \theta}{\cos \theta} = \tan \theta
\]
\[
\frac{\cos \theta}{\sin \theta} = \cot \theta
\]

Set 3: Reciprocal Identities
\[
\sec \theta = \frac{1}{\cos \theta}
\]
\[
\csc \theta = \frac{1}{\sin \theta}
\]
\[
\cot \theta = \frac{1}{\tan \theta}
\]

Set 4: Complementary Angle Identities:
\[
\sin(90^\circ - \theta) = \cos \theta
\]
\[
\cos(90^\circ - \theta) = \sin \theta
\]

Example 1: Assume \(2\sin^2(x) - 5\cos^2(x) = 1\), find the numerical value of \(\sin^2(x)\).

Example 2: Rewrite \(f(x) = 5\csc^2(x) + 7\tan^2(x)\) in terms of \(\cos(x)\).

Example 3: If \(\cos(\theta) = \frac{5}{7}\), find \(\sin(\theta), \tan(\theta), \text{ and } \sec(\theta)\).
Example 4: Assume \( \cos(x) + 6 \sin(x) = \frac{27}{5} \) and \( 6 \cos(x) + \sin(x) = \frac{22}{5} \), find \( \sec(x) \) and \( \tan(x) \).

4. Definition of Trigonometric Functions of any Angles
Place the angle in the standard position and let point \( P(x,y) \) be a point on the terminal side of the angle.

Among the three trig functions: \( \sin(\theta) \), \( \cos(\theta) \), and \( \tan(\theta) \)

Example 5: Find the exact values of the six trigonometric functions of \( \theta \) if \( \theta \) is in standard position and \( P(-4, 7) \) is on the terminal side.

Example 6: Assume \( \theta \) lies in quadrant 3 and the terminal side of \( \theta \) is perpendicular to the line \( y = -18x + 4 \).
**Part 1:** Determine \( \sin(\theta) \)
**Part 2:** Determine \( \sec(\theta) \)
Example 7: A radio transmission tower is 42 feet tall and makes a right angle with the ground. A guy wire is to be attached to the tower 5 feet from the top of the tower and makes an angle of 30 degrees with the ground. Determine the length of the guy wire.

Example 8: The arc shown in the figure is a portion of the unit circle, $x^2 + y^2 = 1$. Express the area of the triangle $\triangle ABP$ as a function of the angle $\theta$. If you need a power of a trig function, such as $\cos^n(\theta)$, write $(\cos(\theta))^n$. The symbol $\theta$ is spelled theta or you can find it in the Mathpad symbol drawer.
Section 5.3 Trigonometric Functions of Real Numbers

1. Definition of Trigonometric Functions of Real Numbers
The value of a trigonometric function at a real number $t$ is its value at the angle of $t$ radians, provided that value exists. So if we place the angle of $t$ radians in the standard position, there will be a point $P$ on the unit circle that corresponds to $t$.

2. Definition of Trigonometric Functions in terms of a Unit Circle
If $t$ is a real number and $P(x,y)$ is the point on the unit circle $U$ that corresponds to $t$, then

Example 1: A point $P(x, y)$ is shown on the unit circle $U$ corresponding to a real number $t$. Find the values of the trigonometric functions at $t$. Assume $a = -12/13$, $b = 5/13$.

Example 2:
Let $P(t)$ be the point on the unit circle $U$ that corresponds to $t$. If $P(t) = (8/17, 15/17)$, find the following.

(a) $P(t + \pi) = (\text{ }, \text{ })$

(b) $P(t - \pi) = (\text{ }, \text{ })$

(c) $P(-t) = (\text{ }, \text{ })$

(d) $P(-t - \pi) = (\text{ }, \text{ })$

Example 3: Use the information that $\sin(t) = \frac{\sqrt{3}}{9}$ and $\frac{\pi}{2} < t < \pi$ to compute $\cos(t)$ and $\tan(t)$. Exact answers only; decimal approximations will be marked incorrect.
Example 4: Find all possible values of \( \sin(t) \) when \( \cos(t) = 4/7 \). If there is more than one answer, enter them in a list separated by commas. Enter an exact expression or a number correct to five decimal places.

**Theorem** on Repeated Function Values for sines and cosines:
If \( n \) is any integer, then
\[
\sin(t + 2\pi n) = \sin t \quad \text{and} \quad \cos(t + 2\pi n) = \cos t
\]

**Formulas for Negatives**
\[
\begin{align*}
\sin(-t) &= -\sin(t), \\
\cos(-t) &= \cos(t), \\
\tan(-t) &= -\tan(t), \\
\csc(-t) &= -\csc(t), \\
\sec(-t) &= \sec(t), \\
\cot(-t) &= -\cot(t).
\end{align*}
\]

**Definition of Periodic Function:** A function is periodic if there exists a positive real number \( k \) such that
\[
f(t + k) = f(t)
\]
for every \( t \) in the domain of \( f \). The least such positive real number \( k \), if it exists, is the period of \( f \).

**Graph of** \( y = \sin(x) \)

**Domain**
Vertical asymptotes
**Range**
\( x \)-intercepts
\( y \)-intercept
**Period**
Even or Odd?
**Symmetry**
Graph of $y = \cos(x)$

Domain
Vertical asymptotes
Range
$x$-intercepts
$y$-intercept
Period
Even or Odd?
Symmetry

Graph of $y = \tan(x)$

Domain
Vertical asymptotes
Range
$x$-intercepts
$y$-intercept
Period
Even or Odd?
Symmetry

Graph of $y = \cot(x)$

Domain
Vertical asymptotes
Range
$x$-intercepts
$y$-intercept
Period
Even or Odd?
Symmetry
**Graph of \( y=\sec(x) \)**

<table>
<thead>
<tr>
<th>Domain</th>
<th>Vertical asymptotes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>x-intercepts</td>
</tr>
<tr>
<td></td>
<td>y-intercept</td>
</tr>
<tr>
<td></td>
<td>Period</td>
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<td></td>
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**Graph of \( y=\csc(x) \)**

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Section 5.4 Values of the Trigonometric Functions
Definition of Reference Angle: Let $\theta$ be a nonquadrantal angle in standard position. The reference angle for $\theta$ is the acute angle $\theta_R$ that the terminal side of $\theta$ makes with the x-axis.

Theorem on Reference Angle: If $\theta$ is a nonquadrantal angle in standard position, then to find the value of a trigonometric function at $\theta$, find its value for the reference angle $\theta_R$ and prefix the appropriate sign.

Example 1. Use reference angle to find the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ if (a) $\theta = \frac{5\pi}{6}$, (b) $\theta = 315^\circ$

Example 2. Refer to the graph of $y = \cos(x)$ to find the exact values of $x$ in the interval $[0, 4\pi]$ that satisfy the equation. Enter solutions in a list separated by commas. Exact answers only: decimal points appearing in your answer will be marked incorrect.)

$$\cos(x) = \frac{1}{2}$$

Example 3: Points on the terminal sides of angles play an important part in the design of arms for robots. Suppose a robot has a straight arm 14 inches long that can rotate about the origin in a coordinate plane. If the robot's hand is located at (14, 0) and then rotates through an angle of $-135^\circ$ what is the new location of the hand?
Example 4. A nonzero, acute angle $\theta \neq 45^\circ$ has terminal side located in quadrant I. All we know is that $\sin(\theta) = \frac{a}{c}$. Some new angles are created using $\theta$. For each new angle, compute the corresponding sine value and select the answer from the drop-down boxes.

Your answer choices:

A: $\frac{a}{c}$

B: $\frac{c}{\sqrt{c^2 - a^2}}$

C: $\frac{c}{\sqrt{c^2 - a^2}}$

D: $\frac{a}{c}$

E: None of the above

(a) New Angle: $\alpha = 720^\circ + \theta$

$\sin(\alpha) =$ [Blank Box]

(b) New Angle: $\beta = -90^\circ + \theta$

$\sin(\beta) =$ [Blank Box]

(c) New Angle: $\mu = -90^\circ + \theta$

$\sin(\mu) =$ [Blank Box]