

# STRONGLY SINGULAR INTEGRALS ALONG CURVES

NORBERTO LAGHI

NEIL LYALL

ABSTRACT. In this article we obtain  $L^2$  bounds for strongly singular integrals along curves in  $\mathbf{R}^d$ ; our results both generalise and extend to higher dimensions those obtained by Chandarana [1] in the plane. Moreover, we show that the operators in question are bounded from  $L \log L$  to weak  $L^1$  at the critical exponent  $\alpha = 0$ .

## 1. INTRODUCTION

It is standard and well known that the Hilbert transform along curves:

$$\mathcal{H}_\gamma f(x) = \text{p.v.} \int_{-1}^1 f(x - \gamma(t)) \frac{dt}{t},$$

is bounded on  $L^p(\mathbf{R}^d)$ , for  $1 < p < \infty$ , where  $\gamma(t)$  is an appropriate curve in  $\mathbf{R}^d$ . In particular, it is known that  $\|\mathcal{H}_\gamma f\|_p \leq C\|f\|_p$ , for  $1 < p < \infty$ , where

$$(1) \quad \gamma(t) = (t, t|t|^k) \text{ or } (t, |t|^{k+1})$$

with  $k \geq 1$ , is a curve in  $\mathbf{R}^2$ . This work was initiated by Fabes and Rivière [5]. The specific result stated above is due to Nagel, Rivière, and Wainger [9]. In [14], Stein and Wainger extended these results to *well-curved*  $\gamma$  in  $\mathbf{R}^d$ ; smooth mappings  $\gamma(t)$  such that  $\gamma(0) = 0$  and

$$\left. \frac{d^k \gamma(t)}{dt^k} \right|_{t=0}, \quad k = 1, 2, \dots$$

span  $\mathbf{R}^d$  (smooth mappings of finite type in a small neighborhood of the origin). For the most recent results and further references, see [3].

It is worth pointing out, however, that  $\mathcal{H}_\gamma$  displays “bad” behavior near  $L^1$ ; Christ [2] showed that  $\mathcal{H}_\gamma$  maps the (parabolic) Hardy space  $H^1$  into weak  $L^1$  for the plane curves  $\gamma(t) = (t, t^2)$ , and furthermore pointed out that  $H^1 \rightarrow L^1$  boundedness cannot hold, while a previous result of Christ and Stein [4] established that  $\mathcal{H}_\gamma$  maps  $L \log L(\mathbf{R}^d)$  into  $L^{1,\infty}(\mathbf{R}^d)$  for a large class of curves  $\gamma$  in  $\mathbf{R}^d$ . Seeger and Tao [12] have shown that  $\mathcal{H}_\gamma$  maps the product Hardy space  $H_{\text{prod}}^1(\mathbf{R}^2)$  into the Lorentz space  $L^{1,2}(\mathbf{R}^2)$ ; the results obtained are sharp, as  $\mathcal{H}_\gamma$  does not map the product Hardy space into any smaller Lorentz space. Finally, the same authors, along with Wright, have shown in [13] that  $\mathcal{H}_\gamma$  maps  $L \log \log L(\mathbf{R}^2)$  into  $L^{1,\infty}(\mathbf{R}^2)$ .

The purpose of this short note is to discuss a strongly singular analogue of these singular integrals along curves  $\gamma(t) = (\gamma_1(t), \dots, \gamma_d(t))$  in  $\mathbf{R}^d$ , namely operators of the form

$$(2) \quad T_\gamma f(x) = \text{p.v.} \int_{-1}^1 H_{\alpha,\beta}(t) f(x - \gamma(t)) dt,$$

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where  $H_{\alpha,\beta}(t) = t^{-1}|t|^{-\alpha}e^{i|t|^{-\beta}}$  is now a strongly singular (convolution) kernel in  $\mathbf{R}$  which enjoys some additional cancellation (note that  $H_{\alpha,\beta}$  is an odd function for  $t \neq 0$ ).

**Theorem 1.1.** *If  $\gamma(t)$  is well-curved, then  $T_\gamma$  is bounded on  $L^2(\mathbf{R}^d)$  if and only if  $\alpha \leq \beta/(d+1)$ .*

Continuing on the work of Zielinski [16], Chandarana [1] obtained the result above for operators of the form (2) in  $\mathbf{R}^2$  along the model homogeneous curves (1). Although Chandarana obtains some partial  $L^p$  results, no endpoint result near  $L^1$  have previously been obtained for the critical value  $\alpha = 0$ ; to that extent we have the following.

**Theorem 1.2.** *If  $\gamma(t)$  is well-curved,  $\alpha = 0$ , and  $\beta > 0$ , then  $T_\gamma : L \log L(\mathbf{R}^d) \rightarrow L^{1,\infty}(\mathbf{R}^d)$ .*

As a consequence of complex interpolation one gets a result involving suitable intermediate spaces, namely

**Corollary 1.3.** *If  $\gamma(t)$  is well-curved, then*

- (i)  $T_\gamma : L^p(\log L)^{2(1/p-1/2)}(\mathbf{R}^d) \rightarrow L^{p,p'}(\mathbf{R}^d)$  whenever  $\alpha p' \leq 2\beta/(d+1)$  and  $1 < p \leq 2$
- (ii)  $T_\gamma : L^{p,p'}(\mathbf{R}^d) \rightarrow (L^p(\log L)^{2(1/2-1/p)}(\mathbf{R}^d))^*$  whenever  $\alpha p \leq 2\beta/(d+1)$  and  $2 \leq p < \infty$ .

Here  $L^{p,q}$  denote the familiar Lorentz spaces, namely

$$L^{p,q}(\mathbf{R}^d) = \left\{ f \text{ measurable on } \mathbf{R}^d : p \int_0^\infty \lambda^{q-1} |\{x : |f(x)| > \lambda\}|^{q/p} d\lambda < \infty \right\},$$

while the  $L^p(\log L)^q$  spaces are defined by

$$L^p(\log L)^q(\mathbf{R}^d) = \left\{ f \text{ measurable on } \mathbf{R}^d : \int_{\mathbf{R}^d} |f(x)|^p \log^q(e + |f(x)|) dx < \infty \right\}.$$

We observe that the statement of Theorem 1.2 is of interest as it bears an element of novelty, namely an endpoint result near  $L^1$ , and as such is more important than the somewhat technical result of Corollary 1.3. We note, however, that while it is simple to prove that  $T_\gamma : L^p(\mathbf{R}^d) \rightarrow L^p(\mathbf{R}^d)$  if  $\alpha(d+1)/\beta < 1 - 2|1/p - 1/2|$  and  $1 < p < \infty$ , Corollary 1.3 provides a first, albeit technical, result for the conjectured sharp range of exponents  $\alpha, \beta$  and  $p$ .

The paper is structured as follows. In the next section we shall perform some standard reductions and prove a basic oscillatory integral estimate. In §3 we complete the proof of Theorem 1.1, while in §4 we give the proof of Theorem 1.2. Finally, in §5 we show how certain estimates found in [13] may be applied in some special two-dimensional cases to obtain better regularity near  $L^1$ .

*Notation.* Throughout this paper,  $C$  shall denote a strictly positive constant whose value may change from line to line and even from step to step, that depends only on the dimension  $d$  and quantities such as  $\alpha$  and  $\beta$  as well as the curve  $\gamma$  in question. Whenever we write  $E = O(F)$  for any two quantities  $E$  and  $F$  we shall mean that  $|E| \leq C|F|$ , for some strictly positive constant  $C$ .

## 2. $L^2$ REGULARITY AND A LEMMA OF VAN DER CORPUT TYPE

We first focus our attention on  $L^2$  estimates. We shall dyadically decompose our operator  $T_\gamma$  in the standard way. To this end we let  $\eta(t) \in C_0^\infty(\mathbf{R}_+)$  be so that  $\eta \equiv 1$  if  $0 \leq t \leq 1$ , and

$\eta \equiv 0$  if  $t \geq 2$ , then we let  $\vartheta(t) = \eta(t) - \eta(2t)$ , so that  $\sum_{j \in \mathbf{Z}} \vartheta(2^j t) \equiv 1$  for  $t > 0$ . We then consider the rescaled operators

$$(3) \quad T_j f(x) = 2^{j\alpha} \int \vartheta(t) t^{-1} |t|^{-\alpha} e^{i2^{j\beta}|t|^{-\beta}} f(x - \gamma(2^{-j}t)) dt,$$

where, of course,  $\text{supp } \vartheta \subset \{t : 1/2 \leq |t| \leq 2\}$ . Theorem 1.1 will then be a consequence of the following two results (together with an application of Cotlar's lemma and a standard limiting argument).

**Theorem 2.1** (Dyadic Estimate). *If  $\gamma$  satisfies the finite type condition of Theorem 1.1, then*

$$\|T_j f\|_{L^2(\mathbf{R}^d)} \leq C 2^{j(\alpha - \beta/(d+1))} \|f\|_{L^2(\mathbf{R}^d)}.$$

**Proposition 2.2** (Almost Orthogonality). *If  $\gamma$  satisfies the finite type condition of Theorem 1.1 and  $\alpha \leq \beta/(d+1)$ , then the dyadic operators satisfy the estimate*

$$\|T_j^* T_{j'}\|_{L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)} + \|T_{j'} T_j^*\|_{L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)} \leq C 2^{-\delta|j'-j|},$$

for some  $\delta > 0$ .

Key to the proofs of Theorem 2.1 and Proposition 2.2, which we shall give in the next section, is the following result which is an immediate consequence (of the proof) of a lemma of Ricci and Stein [10], see also [14].

**Lemma 2.3.** *Let*

$$\varphi(t) = t^{b_0} + \mu_1 t^{b_1} + \dots + \mu_n t^{b_n}$$

be a real-valued function, with  $\mu_1, \dots, \mu_n$  arbitrary real parameters,  $a$  and  $b$  real constants which satisfy  $0 < c \leq a < b \leq c^{-1}$ , and  $b_0, b_1, \dots, b_n$  distinct nonzero real exponents, then

$$(4) \quad \left| \int_a^b e^{i\lambda\varphi(t)} dt \right| \leq C \lambda^{-1/(n+1)},$$

where  $C$  does not depend on  $\mu_1, \dots, \mu_n$  or  $\lambda$ .

Ricci and Stein in fact proved that if  $b_0, b_1, \dots, b_n$  are distinct *positive* real exponents, then

$$(5) \quad \left| \int_a^b e^{i\lambda\varphi(t)} dt \right| \leq C \lambda^{\min\{1/b_0, 1/(n+1)\}}$$

uniformly in  $0 \leq a < b \leq 1$ .

The analogue of (4) and (5) where a cutoff function of bounded variation is inserted in the amplitude of the integral follows immediately from a standard integration by parts argument.

The proof of Lemma 2.3 is essentially just that of Ricci and Stein, but we shall outline the argument here. First we recall a standard formulation of van der Corput's lemma; see [11].

**Proposition 2.4** (Van der Corput). *Suppose  $\psi$  is a function in  $C^k([a, b])$  which satisfies the estimate  $|\psi^{(k)}(x)| \geq C > 0$  for all  $x \in (a, b)$ , then*

$$\left| \int_a^b e^{i\lambda\psi(t)} dt \right| \leq k C_k \lambda^{-1/k},$$

whenever (i)  $k = 1$  and  $\psi''(x)$  has at most one zero, or (ii)  $k \geq 2$ .

In light of Proposition 2.4 we see that Lemma 2.3 will be a consequence of the following.

**Lemma 2.5.** *There exists a constant  $C_1 = C_1(b_0, b_1, \dots, b_n)$  independent of  $\mu_1, \dots, \mu_d$  and  $\lambda$  so that for each  $t \in [a, b]$  we have that  $|\varphi^{(k)}(t)| \geq C_1 t^{b_0 - k}$  for at least one  $k = 1, \dots, n + 1$ .*

More precisely, in order to prove Lemma 2.3 we split the interval  $[a, b]$  into a finite number of subintervals in such a way that one of the inequalities of Lemma 2.5 holds on each; if the first of the inequalities holds one can further split into intervals where  $\varphi'(t)$  is monotonic. The number of subintervals depends only on  $n$  and the desired conclusion follows from Proposition 2.4 (and the fact that  $a$  and  $b$  are contained in a compact subinterval of  $(0, \infty)$ ).

*Proof of Lemma 2.5.* Observe that if we set  $\mu_0 = 1$ , then for  $k = 1, \dots, n + 1$ ,

$$t^{-b_0+k} \varphi^{(k)}(t) = \sum_{j=1}^{n+1} m_{k,j} \mu_{j-1} t^{b_{j-1}-b_0},$$

where  $m_{k,j} = \prod_{i=1}^k (b_{j-1} - i + 1)$ .

If we now define  $w = (w_1, \dots, w_{n+1})$  with  $w_k = t^{-b_0+k} \varphi^{(k)}(t)$  and  $v = (v_1, \dots, v_{n+1})$  with  $v_i = \mu_{i-1} t^{b_{i-1}-b_0}$ , then we have  $w = Mv$ , where  $M$  is a Vandermonde matrix with

$$\det M = \prod_{j=0}^n b_j \prod_{0 \leq i < j \leq n} (b_i - b_j). \quad \square$$

### 3. THE PROOFS OF THEOREM 2.1 AND PROPOSITION 2.2

Recall that establishing  $L^2$  estimates for the dyadic operators  $T_j$  is equivalent to establishing *uniform* bounds, in  $\mathbf{R}^d$ , for the multipliers

$$(6) \quad m_j(\xi) = 2^{j\alpha} \int \vartheta(t) t^{-1} |t|^{-\alpha} e^{i\psi(t)} dt,$$

where  $\psi(t) = 2^{j\beta} |t|^{-\beta} - \gamma(2^{-j}t) \cdot \xi$ . We shall take this *multiplier approach* to prove both Theorem 2.1 and Proposition 2.2.

It follows from the proposition below that we may, with no loss in generality, assume that our curves  $\gamma(t)$  are of *standard type*; that is approximately homogeneous, taking the form

$$(7) \quad \gamma_k(t) = \frac{t^{a_k}}{a_k!} + \text{higher order terms}$$

for  $k = 1, \dots, d$  with  $1 \leq a_1 < \dots < a_d$ .

**Proposition 3.1.** *To every smooth well-curved  $\gamma(t)$  there exists a constant nonsingular matrix  $M$  such that  $\tilde{\gamma}(t) = M\gamma(t)$ , is of standard type.*

For a simple proof of this result, see [14].

We note that in the model case corresponding to the homogeneous (monomial) curves of the form  $\gamma_k(t) = t^{a_k}$ , we may write  $\psi(t) = 2^{j\beta} \varphi(t)$ , where

$$\varphi(t) = |t|^{-\beta} - (\mu_1 t^{a_1} + \dots + \mu_d t^{a_d}),$$

with

$$\mu = 2^{-j} \circ_{\beta} \xi = (2^{-j(\beta+a_1)} \xi_1, \dots, 2^{-j(\beta+a_d)} \xi_d).$$

In addition to observing the natural manner in which the nonisotropic dilations above have entered into the analysis of this problem we also point out that Theorem 2.1 in fact now

follows immediately from Lemma 2.3 in this model case. In fact by continuity we also obtain the estimates

$$(8) \quad |m_j(\xi)| \leq C2^{j(\alpha-\beta/(d+1))}$$

for standard type curves (7) provided that the parameter  $\mu$  remains bounded.

Thus, in order to establish Theorem 2.1 for standard type curves we must obtain uniform multiplier bounds of the form (8) or better for all large  $|\mu|$ . The following key result achieves exactly what we need to prove Theorem 2.1 for standard type curves and some, the additional savings are used in a crucial way in the proof of Proposition 2.2.

**Proposition 3.2** (Refined Dyadic Estimate). *If  $\gamma(t)$  is a curve of standard type, then*

(i) *for all  $\xi \in \mathbf{R}^d$*

$$|m_j(\xi)| \leq C2^{j(\alpha-\beta/(d+1))}(1 + |2^{-j} \circ_\beta \xi|)^{-1/(d+1)}$$

(ii) *there exists  $\varepsilon > 0$  fixed, such that if  $|2^{-j} \circ_\beta \xi| \notin (\varepsilon, \varepsilon^{-1})$ , then*

$$|m_j(\xi)| \leq C2^{j(\alpha-\beta/d)}(1 + |2^{-j} \circ_\beta \xi|)^{-1/d}$$

*with  $\varepsilon$  (and  $C$ ) independent of both  $j$  and  $\xi$ .*

*Proof.* We modify our approach above and write  $\psi(t) = \pm 2^{j\beta} \max\{1, |2^{-j} \circ_\beta \xi|\} \varphi(t)$ , with

$$\varphi(t) = t^{b_0} + \mu_1 t^{b_1} + \dots + \mu_d t^{b_d},$$

and

$$b_0 = \begin{cases} -\beta & \text{if } \max_k \{2^{-j(\beta+a_k)} |\xi_k|\} \leq 1 \\ \ell & \text{if } 2^{-j(\beta+a_\ell)} |\xi_\ell| = \max_k \{2^{-j(\beta+a_k)} |\xi_k|\} \geq 1 \end{cases}.$$

It then follows immediately that  $|\mu_k| \leq 1$  for all  $k = 1, \dots, d$ , and by continuity we obtain the estimate

$$(9) \quad |m_j(\xi)| \leq C2^{j(\alpha-\beta/(d+1))}(1 + |2^{-j} \circ_\beta \xi|)^{-1/(d+1)}$$

for all curves of standard type.

Moreover, it is clear that there exists  $\varepsilon > 0$  such that if  $|2^{-j} \circ_\beta \xi| \leq \varepsilon$ , then

$$|m_j(\xi)| \leq C2^{j(\alpha-\beta)}.$$

Now if instead we assume that  $|2^{-j} \circ_\beta \xi| \geq \varepsilon^{-1}$  for some  $\varepsilon > 0$ , then we may choose a  $k$  such that

$$2^{-j(b_k+\beta)} |\xi_k| \geq \varepsilon^{-(b_k+\beta)} \quad \text{and} \quad 2^{-j b_k} |\xi_k| \geq 2^{-j b_i} |\xi_i|$$

for all  $i \neq k$ . It then follows from Lemma 2 of [10] (the analogue of Lemma 2.5 in that setting) that if

$$\Phi(t) = \sum_{i=1}^d \mu_i t^{b_i},$$

with  $\mu_i = 2^{-j(b_i-b_k)} \xi_i / \xi_k$ , then  $|\Phi^{(\ell)}(t)| \geq C t^{b_k-\ell}$  for some  $\ell = 1, \dots, d$ .

It then follows from the fact that  $2^{-j(b_k+\beta)} |\xi_k| \geq \varepsilon^{-(b_k+\beta)}$  and  $|\mu_i| \leq 1$  for all  $i = 1, \dots, d$ , that

$$\left| \int \vartheta(t) t^{-1} |t|^{-\alpha} e^{i[2^{j\beta}|t|^{-\beta} - \gamma(2^{-j}t) \cdot \xi]} dt \right| \leq C 2^{j b_k/d} |\xi_k|^{-1/d}$$

provided  $\varepsilon > 0$  is chosen small enough (and  $j$  large enough).  $\square$

*Proof of Proposition 2.2.* We shall only establish the desired estimate for  $T_j^*T_{j'}$ ; the proof of the other estimate is analogous.

It follows from Theorem 2.1 that the operators  $T_j$  are uniformly bounded on  $L^2(\mathbf{R}^d)$  whenever  $\alpha \leq \beta/(d+1)$ , and since we also have that  $T_j^*T_{j'}f(x) = \overline{K}_j(-\cdot) * K_{j'} * f(x)$ , where  $\widehat{\overline{K}_j}(\xi) = m_j(\xi)$ , we observe that

$$(10) \quad \|T_j^*T_{j'}\| = \|\overline{m}_j(\xi)m_{j'}(\xi)\|_{L^\infty},$$

and we can clearly assume that  $|j' - j| \gg 1$ .

Let  $\varepsilon > 0$  be the constant given in Proposition 3.2 and without loss in generality we assume that  $j' \geq j + C_0$ , where  $2^{C_0(\beta+a_1)} \gg \varepsilon^{-2}$ . We now distinguish between two cases.

(i) If  $|2^{-j'} \circ_\beta \xi| \leq \varepsilon$ , it then follows from Proposition 3.2 that

$$|m_{j'}(\xi)| \leq C2^{j'(\alpha-N\beta)} \leq C2^{-j'N'\beta} \leq C2^{-(j'-j)N'\beta},$$

for all  $N' > 0$ .

(ii) If  $|2^{-j'} \circ_\beta \xi| > \varepsilon$ , then  $|2^{-j} \circ_\beta \xi| \geq C2^{C_0(\beta+a_1)}\varepsilon \geq \varepsilon^{-1}$ , and appealing to Proposition 3.2 once more it follows that

$$\begin{aligned} |m_j(\xi)| &\leq C2^{j(\alpha-\beta/d)}|2^{-j} \circ_\beta \xi|^{-1/d} \\ &\leq C|2^{j'-j}2^{-j'} \circ_\beta \xi|^{-1/d} \\ &\leq C2^{-(j'-j)(\beta+a_1)/d}|2^{-j'} \circ_\beta \xi|^{-1/d} \\ &\leq C\varepsilon^{-1/d}2^{-(j'-j)(\beta+a_1)/d} \end{aligned}$$

The result then follows from estimate (10) and Theorem 2.1.  $\square$

We finally comment on the necessity of the condition  $\alpha \leq \beta/(d+1)$  in the statement of Theorem 1.1. It is not too difficult to see that if we consider the dyadic operator  $T_j$  along the curve

$$\gamma(t) = (t^{a_1}, \dots, t^{a_d})$$

with  $1 \leq a_1 < \dots < a_d$ , it is possible to find constants  $c_1, \dots, c_d$  such that the relevant multiplier  $m_j = m_j(\xi)$  satisfies

$$A2^{j(\alpha-\beta/(d+1))} \leq \left| m_j \left( c_1\xi_1, c_2\xi_1^{\frac{\beta+a_2}{\beta+a_1}}, \dots, c_d\xi_1^{\frac{\beta+a_d}{\beta+a_1}} \right) \right| \leq \frac{1}{A} 2^{j(\alpha-\beta/(d+1))}$$

for some absolute constant  $0 < A < 1$ , and as such the result is sharp.

#### 4. ESTIMATES NEAR $L^1$

We now turn our attention to the proof of Theorem 1.2. It relies on the result obtained by Christ and Stein in [4]; indeed, we shall show that the general statement proven by these two authors applies to the operator (2).

Before proceeding, we fix some notation. For any tempered distribution  $u \in \mathcal{S}'(\mathbf{R}^d)$  we indicate by  $u^{x_0}$  its translate by  $x_0$ , namely

$$\langle u^{x_0}(x), \phi(x) \rangle = \langle u(x), \phi(x - x_0) \rangle$$

for all test functions  $\phi$ . Moreover, the  $(p, q)$  convolution norm of the operator given by convolution with  $u$  is defined to be

$$\|u\|_{CV(p,q)} = \sup_{f \in L^p} \|f * u\|_{L^q} / \|f\|_{L^p}.$$

We now summarize the assumptions of the Christ-Stein theorem. Let  $Tf(x) = f * K(x)$  be a convolution operator, where  $K$  is a tempered distribution. Now, consider the nonisotropic dilations

$$x \mapsto r \circ x = (r^{a_1}x_1, \dots, r^{a_d}x_d);$$

if  $\rho(x)$  is defined to be the unique  $r > 0$  so that  $|r^{-1} \circ x| = 1$ , then  $\rho$  becomes a quasi-norm homogenous with respect to the dilations above, see [14]. Thus, we may define the distributions

$$(11) \quad K_j(x) = \vartheta(2^j \rho(x))K(x).$$

**Theorem 4.1** (Christ-Stein [4]). *Suppose  $T = \sum_{j \in \mathbf{Z}} T_j$ , where  $T_j f(x) = f * K_j(x)$  as defined above. Assume that there exist some constants  $\delta, \varepsilon > 0$  so that*

- (i)  $\|K_{j+\ell} - K_{j+\ell}^{x_0}\|_{CV(2,2)} \leq C2^{-\varepsilon\ell}$  for all  $y$  with  $\rho(y) \leq C2^j$  and all  $j \in \mathbf{Z}$ ,  $\ell \in \mathbf{Z}_+$
- (ii)  $\|K_j\|_{L^1} \leq C$  uniformly in  $j$
- (iii)  $\|T_j T_{j'}^*\|_{L^2 \rightarrow L^2} + \|T_j^* T_{j'}\|_{L^2 \rightarrow L^2} \leq C2^{-\delta|j-j'|}$  for all  $j, j' \in \mathbf{Z}$ .

Then  $T : L \log L(B) \rightarrow L^{1,\infty}(B)$  for any bounded set  $B \subset \mathbf{R}^d$ .

The above statement provides a local regularity result; however, since we are dealing with an operator given by convolution with a compactly supported kernel, one may actually use the Christ-Stein theorem to obtain a global result.

In order to see how the Christ-Stein theorem applies to the operator  $T_\gamma$  in (2) when  $\alpha = 0$ , we first consider the model case  $\gamma(t) = (t^{a_1}, \dots, t^{a_d})$ , where the  $a_j$  are distinct positive integers and note that the kernel  $K_\gamma$  of  $T_\gamma$  may be written as

$$K_\gamma(x) = \iint e^{i|t|^{-\beta} + \xi \cdot (x_1 - t^{a_1}, \dots, x_d - t^{a_d})} \chi(t) t^{-1} dt d\xi.$$

If we now define, for each  $j \geq 0$ ,

$$K_{\gamma,j} = \vartheta(2^j \rho(x))K_\gamma(x)$$

as in (11), then it is simple to see that for a test function  $f$  one has

$$\langle K_{\gamma,j}, f \rangle = \int e^{i|t|^{-\beta}} \chi(t) t^{-1} \vartheta(\rho(2^j \circ (t^{a_1}, \dots, t^{a_d}))) f(t^{a_1}, \dots, t^{a_d}) dt$$

and as such

$$T_{\gamma,j} f(x) = \int e^{i2^{j\beta}|t|^{-\beta}} \vartheta(\rho(\gamma(t))) t^{-1} f(x - \gamma(t)) dt.$$

It is therefore clear that the operators  $T_{\gamma,j}$  are nearly identical to the operators  $T_j$  in (3); the cutoff function found in the definition of the kernels  $K_{\gamma,j}$  still restricts the  $t$  variable to the set where  $|t| \approx 1$ . Note that trivially

$$\|K_{\gamma,j}\|_{L^1} \leq C$$

and

$$\|T_{\gamma,j} T_{\gamma,j'}^*\|_{L^2 \rightarrow L^2} + \|T_{\gamma,j}^* T_{\gamma,j'}\|_{L^2 \rightarrow L^2} \leq C2^{-\varepsilon|j-j'|}$$

for all  $j, j' \in \mathbf{Z}_+$  as the almost orthogonality of the operators  $T_{\gamma, j}$  is truly equivalent to that of the operators  $T_j$ , and this has been proven in the previous section.

Thus, in order to apply the Christ-Stein result we need only show that

$$\|K_{\gamma, j+\ell} - K_{\gamma, j+\ell}^{x_0}\|_{CV(2,2)} \leq C2^{-\varepsilon\ell}$$

for all  $\ell \in \mathbf{Z}_+$  and some  $\varepsilon > 0$ ; note that  $j + \ell \geq 0$ , otherwise the kernel is identically vanishing. To verify this condition it suffices to just check that

$$(12) \quad \|\widehat{K_{\gamma, j+\ell}}(\xi) - \widehat{K_{\gamma, j+\ell}^{x_0}}(\xi)\|_{L^\infty} = \|(1 - e^{ix_0 \cdot \xi})\widehat{K_{\gamma, j+\ell}}\|_{L^\infty} \leq C2^{-\varepsilon\ell},$$

where

$$\widehat{K_{\gamma, j+\ell}}(\xi) = \int e^{i[|t|^{-\beta} + \xi \cdot (t^{a_1}, \dots, t^{a_d})]} \vartheta(2^{j+\ell} \rho(\gamma(t))) t^{-1} dt.$$

First of all, note that if  $j \geq 0$ , there is nothing to show, as

$$|\widehat{K_{\gamma, j+\ell}}(\xi)| \leq C2^{-(j+\ell)\beta/(d+1)}.$$

However, this pointwise estimate also shows that if  $j < 0$ , but  $|j| < (1 - \delta)\ell$  for some  $\delta > 0$ , then (12) is also verified. To deal with the remaining case  $j < 0, |j| > (1 - \delta)\ell$ , we note that

$$\left| (1 - e^{ix_0 \cdot \xi})\widehat{K_{\gamma, j+\ell}}(\xi) \right| \leq C \left| \widehat{K_{\gamma, j+\ell}}(\xi) \right| \min\{1, |\xi \cdot x_0|\}.$$

Since  $|x_0| \leq C2^j$ , problems may arise only if  $2^{j(1-\delta)} \ll |\xi| \leq C$ ; indeed, if  $|\xi| \leq C2^{j(1-\delta)}$ , the bound

$$\left| (1 - e^{ix_0 \cdot \xi})\widehat{K_{\gamma, j+\ell}}(\xi) \right| \leq C2^{\delta j} \leq C2^{-\delta\ell/2}$$

holds. Thus, consider the case  $|\xi| \gg 2^{j(1-\delta)}$ ; here we may use estimate (9) to get

$$\left| \widehat{K_{\gamma, j+\ell}}(\xi) \right| \leq C2^{-(j+\ell)\beta/(d+1)} (1 + |2^{-(j+\ell)} \circ_\beta \xi|)^{-1/(d+1)}.$$

Using that  $j < 0, |j| > (1 - \delta)\ell$ , and the size of  $|\xi|$ , one obtains the estimate that

$$(1 + |2^{-(j+\ell)} \circ_\beta \xi|)^{-1/(d+1)} \leq C2^{-\ell/4(d+1)},$$

provided  $\delta > 0$  is sufficiently small. This is enough to prove (12) in this case and Theorem 1.2 for model case curves.

Passing to the general case of standard curves is not difficult. If  $\gamma$  is a curve of standard type, then we again define

$$T_{\gamma, j} f(x) = \int e^{i|t|^{-\beta}} \chi(t) t^{-1} \vartheta(\rho(2^j \circ \gamma(t))) f(x - \gamma(t)) dt,$$

with  $\rho$  homogeneous with respect to the dilations

$$r \circ x = (r^{a_1} x_1, \dots, r^{a_d} x_d).$$

Since  $\gamma$  is approximately homogeneous with respect to the same dilations, we see that the Fourier transform of the kernel  $K_{\gamma, j}$  is given by

$$\widehat{K_{\gamma, j}}(\xi) = \int e^{i[2^{j\beta}|t|^{-\beta} + \xi \cdot \gamma(2^{-j}t)]} \vartheta(\rho(t^{a_1} + O(2^{-j}), \dots, t^{a_d} + O(2^{-j}))) dt.$$

Now, note that for all  $j > 0$  sufficiently large the cutoff function in the definition of  $m_j$  restricts  $t$  to the set where  $|t| \approx 1$ , and has uniformly bounded  $C^\infty$  seminorms. Thus, the estimate of

<sup>1</sup> We shall no longer make explicit mention of the fact that the kernels  $K_j$  have the same properties as the kernels defining the operators  $T_j$  in (3).



Lemma 2.3 applies, implying estimate (9), while the almost orthogonality of the operators  $T_{\gamma,j}$  may be obtained as in Proposition 2.2.  $\square$

To prove Corollary 1.3 one may form an analytic family of operators in the standard way and proceed as in [1]; then, the appropriate version of Stein's interpolation theorem applies. We omit the details.

## 5. ESTIMATES IN TWO DIMENSIONS

In [13] a very interesting regularity result (near  $L^1$ ) for singular Radon transforms was proven. To describe it, let  $\Sigma$  be a hypersurface in  $\mathbf{R}^d$  and let  $\mu$  be a compactly supported smooth density on  $\Sigma$ , i.e.  $\mu = \vartheta(x)d\sigma$  where  $\vartheta \in C_0^\infty(\mathbf{R}^d)$  and  $d\sigma$  is surface measure on  $\Sigma$ . Let  $\mu_j$  be dilates of  $\mu$  defined by

$$\langle \mu_j, f \rangle = \langle \mu, f(2^j \circ \cdot) \rangle,$$

where  $\circ$  denotes the nonisotropic dilations introduced in §4. Consider the singular Radon transform

$$\mathcal{R}f(x) = \sum_{j \in \mathbf{Z}} \mu_j * f(x).$$

Under the assumption that the Gaussian curvature of  $\Sigma$  does not vanish to infinite order at any point (in  $\Sigma$ ) and that the cancellation condition

$$\int d\mu = 0$$

holds, Seeger, Tao and Wright showed that

$$\mathcal{R} : L \log \log L(\mathbf{R}^d) \rightarrow L^{1,\infty}(\mathbf{R}^d).$$

It is not difficult to see that the local version  $\mathcal{R}_{\text{loc}}f(x) = \sum_{k < C} \mu_k * f(x)$  is also of weak type  $L \log \log L$ .

We wish to apply this latest result to the operator  $T_\gamma$  in (2) in the case  $d = 2$ ,  $\alpha = 0$ ; moreover, we choose  $\gamma$  to have the special form  $(t, t|t|^b)$ ,  $b > 0$ . In order to do so we now choose a smooth cutoff function  $\vartheta = \vartheta(t)$  supported in  $[1/2, 1]$  with the property that  $\sum_{j \in \mathbf{Z}_+} \vartheta(2^j t) \equiv 1$  for, say,  $0 < t \leq 1/2$ ; moreover, we choose another smooth cutoff  $\eta$  with the property that  $\eta(t) \equiv 1$  for  $|t| \leq M$  and  $\eta(t) \equiv 0$  for  $|t| > 2M$ , where  $M \gg 1$ . Thus, if we pick the measure  $\mu$  to be

$$\mu(x) = e^{i|x_1|^{-\beta}} x_1^{-1} \vartheta(|x_1|) \eta(|x_2|),$$

we see that its action on test function  $\phi$  is given by

$$\langle \mu, \phi \rangle = \int e^{i|t|^{-\beta}} t^{-1} \vartheta(|t|) \eta(t|t|^b) \phi(t, t|t|^b) dt = \int e^{i|t|^{-\beta}} t^{-1} \vartheta(|t|) \phi(t, t|t|^b) dt,$$

if we choose the number  $M$  in the definition of  $\eta$  to be large enough. Further, it is simple to see that now  $\int d\mu = 0$  and that the curvature of  $\gamma$  does not vanish to infinite order on  $[1/2, 1]$ .

Now, if we choose nonisotropic dilations

$$r \circ x = (rx_1, r^{b+1}x_2),$$

it is simple to see that

$$T_\gamma f(x) = \sum_{j \in \mathbf{Z}_+} \mu_j * f(x)$$

and the result in [13] gives the following.

**Theorem 5.1.** *Let  $d = 2$  and  $\gamma(t) = (t, t|t|^b)$ ,  $b > 0$ . If  $\alpha = 0$ , then*

$$T_\gamma : L \log \log L(\mathbf{R}^2) \rightarrow L^{1,\infty}(\mathbf{R}^2).$$

If we interpolate this estimate with the sharp  $L^2$  bounds of Theorem 1.1 we get a better regularity result (in this special two dimensional case) than the one provided by Corollary 1.3. The precise statement can be obtained by utalizing the same procedure as in Corollary 1.3.

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SCHOOL OF MATHEMATICS AND MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, THE UNIVERSITY OF EDINBURGH, JCM BUILDING, THE KING’S BUILDINGS, EDINBURGH EH9 3JZ, UNITED KINGDOM

*E-mail address:* N.Laghi@ed.ac.uk

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF GEORGIA, BOYD GRADUATE STUDIES RESEARCH CENTER, ATHENS, GA 30602, USA

*E-mail address:* lyall@math.uga.edu