

STRONGLY SINGULAR CONVOLUTION OPERATORS ON THE HEISENBERG GROUP

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ABSTRACT. We consider the L^2 mapping properties of a model class of strongly singular integral operators on the Heisenberg group \mathbf{H}^n ; these are convolution operators on \mathbf{H}^n whose kernels are too singular at the origin to be of Calderón-Zygmund type. This strong singularity is compensated for by introducing a suitably large oscillation. Our results are obtained by utilizing the group Fourier transform and uniform asymptotic forms for Laguerre functions due to Erdélyi .

OVERTURE: STRONGLY SINGULAR CONVOLUTION OPERATORS ON \mathbf{R}^d

These are convolution operators whose kernels are too singular at the origin to be of Calderón-Zygmund type. We choose to compensate for this strong singularity by introducing a suitably large oscillation. More precisely these are the operators given formally by

$$Sf = f * K,$$

where K be a distribution on \mathbf{R}^d that away from the origin agrees with the function

$$K(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|),$$

where $\beta > 0$ and χ is smooth cut off function which equals one near the origin¹. Operators of this type were first studied by Hirschman [7] in the case $d = 1$ and then in higher dimensions by Wainger [12], Fefferman [3], and Fefferman and Stein [4]. What is of interest here is the precise relationship between the size of the singularity and that of the required oscillation in order for S to extend to a bounded operator on $L^2(\mathbf{R}^d)$.

Theorem A. *S extends to a bounded operator on $L^2(\mathbf{R}^d)$ if and only if $\alpha \leq \frac{d\beta}{2}$.*

Sketch of proof. Since S is translation invariant it may be realized as a Fourier multiplier,

$$\widehat{Sf}(\xi) = \widehat{f}(\xi) \cdot m(\xi),$$

where $\widehat{}$ denotes the Fourier transform and $m = \widehat{K}$, the fact that K is a compactly supported distribution ensures that $m(\xi)$ is a function. Plancherel's theorem then implies that

$$\|Sf\|_{L^2(\mathbf{R}^d)} \leq A\|f\|_{L^2(\mathbf{R}^d)} \quad \text{if and only if} \quad |m(\xi)| \leq A, \text{ uniformly in } \xi.$$

Since K is also radial, i.e. $K(x) = K_0(x)$ for some function K_0 , we have

$$m(\xi) = (2\pi)^{\frac{d}{2}} \int_0^\infty K_0(r) J_{\frac{d-2}{2}}(r|\xi|) (r|\xi|)^{\frac{2-d}{2}} r^{d-1} dr,$$

¹ The distribution-valued function $\alpha \mapsto K$, initially defined for $\text{Re } \alpha < 0$, continues analytically to all of \mathbf{C} .

where $J_{\frac{d-2}{2}}$ is a Bessel function; see [10]. Using the well known asymptotics of Bessel functions it is then straightforward to show that $m(\xi)$ remains bounded for small $|\xi|$ and that for large $|\xi|$,

$$(1) \quad m(\xi) = c_1 |\xi|^{\frac{\alpha - \frac{d\beta}{2}}{\beta+1}} e^{ic_2 |\xi|^{\frac{\beta}{\beta+1}}} + O(|\xi|^{\frac{\alpha - \frac{d+1}{2}\beta}{\beta+1}}). \quad \square$$

Remark. One can establish Theorem A independent of Fourier transform methods, see [8].

This article is devoted to the study of analogous operators on the Heisenberg group.

1. INTRODUCTION

The Heisenberg group \mathbf{H}^n is of course $\mathbf{C}^n \times \mathbf{R}$ endowed with the group law

$$[z, t] \cdot [w, s] = [z + w, t + s + \frac{1}{2} \operatorname{Im}(z \cdot \bar{w})],$$

with identity the origin and inverses given by $[z, t]^{-1} = [-z, -t]$. The following transformations are automorphisms of the group \mathbf{H}^n :

- the *nonisotropic dilations* $[z, t] \mapsto \delta \circ [z, t] = [\delta z, \delta^2 t]$, for all $\delta > 0$;
- the *rotations* $[z, t] \mapsto [Uz, t]$, with U a unitary transformation of \mathbf{C}^n .

We define the “norm” function ρ on \mathbf{H}^n to be

$$\rho(z, t) = (|z|^4 + t^2)^{1/4},$$

and note that it is smooth, invariant under the rotations defined above, and homogeneous with respect to the automorphic dilations. The usual Lebesgue measure $dz dt$ on $\mathbf{C}^n \times \mathbf{R}$ is the Haar measure for \mathbf{H}^n .

We define our strongly singular kernels on \mathbf{H}^n to be the distributions K that agree, for $(z, t) \neq (0, 0)$, with the functions

$$K(z, t) = \rho(z, t)^{-Q-\alpha} e^{i\rho(z, t)^{-\beta}} \chi(\rho(z, t)) \text{ with } \beta > 0,$$

where $Q = 2n + 2$ is the homogeneous dimension of \mathbf{H}^n and χ is defined as before².

Our main result is then the following.

Theorem 1.1. *The operators T on \mathbf{H}^n , formally given by $Tf = f * K$, satisfy the following:*

- (i) *If $\alpha \leq n\beta$, then T extends to a bounded operator from $L^2(\mathbf{H}^n)$ to itself.*
- (ii) *If T is a bounded operator on $L^2(\mathbf{H}^n)$ then $\alpha \leq (n + \frac{1}{2})\beta$.*

Remark 1.2. There is a “gap of $\frac{1}{2}$ ” between necessary and sufficient conditions. Note also that the necessary condition coincides with the Euclidean condition $\alpha \leq \frac{2n+1}{2}\beta$.

² As in the Euclidean case the distribution-valued function $\alpha \mapsto K$, initially defined for $\operatorname{Re} \alpha < 0$, continues analytically to the entire complex plane.

2. REDUCTION TO LAGUERRE TRANSFORM ESTIMATES

Let $\epsilon > 0$ and set

$$K^\epsilon(z, t) = e^{-\epsilon\rho(z,t)^{-\beta}} K(z, t)$$

and for $f \in L^2(\mathbf{H}^n)$ we define

$$T^\epsilon f(z, t) = f * K^\epsilon(z, t),$$

where convolution is taken with respect to the group structure on \mathbf{H}^n . It is then easy to see that if, for fixed $\epsilon > 0$, we integrate the function K^ϵ by parts N times and take the limit as $\epsilon \rightarrow 0$ then this must agree with the unique analytic continuation of K to the half plane $\text{Re}(\alpha) < N\beta$. It then follows that for $f \in \mathcal{S}(\mathbf{H}^n)$,

$$Tf(z, t) = \lim_{\epsilon \rightarrow 0} T^\epsilon f(z, t).$$

We shall therefore, in the following, content ourselves with studying the operator T^ϵ .

2.1. Group Fourier transform. Our operator T^ϵ on \mathbf{H}^n can be realized on the “transform” side, via the group Fourier transform, as a multiplication of operators,

$$\widehat{T^\epsilon f}(\lambda) = \widehat{f}(\lambda) \cdot M(\lambda),$$

where $M(\lambda)$ is the group Fourier transform of K^ϵ . Recall that, for each $\lambda \neq 0$, $M(\lambda)$ is an operator on the Hilbert space $L^2(\mathbf{R}^n)$ given by

$$M(\lambda)\varphi(\xi) = \int_{\mathbf{H}^n} K^\epsilon(z, t)\pi_\lambda(z, t)\varphi(\xi)dzdt,$$

where the π_λ , given for $z = x + iy$ by

$$\pi_\lambda(z, t)\varphi(\xi) = e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y + t)}\varphi(\xi + y)$$

are the infinite dimensional irreducible unitary representations of \mathbf{H}^n , the so called Schrödinger representations. It then follows from Plancherel’s theorem for the group Fourier transform that

$$\|T^\epsilon f\|_{L^2(\mathbf{H}^n)} \leq A\|f\|_{L^2(\mathbf{H}^n)} \quad \text{if and only if} \quad \|M(\lambda)\|_{Op} \leq A \text{ uniformly over } \lambda \neq 0.$$

Now since K^ϵ were chosen *radial* on \mathbf{H}^n , i.e. $K^\epsilon(z, t) = K_0^\epsilon(|z|, t)$ for some function K_0^ϵ , then it is a well known result of Geller [6] that the operators $M(\lambda)$ are in fact, for each $\lambda \neq 0$, diagonal with respect to a (rescaled) Hermite basis for $L^2(\mathbf{R}^n)$. More precisely

$$M(\lambda) = C_n(\delta_{\mathbf{j}, \mathbf{k}} \mu(|\mathbf{k}|, \lambda))_{\mathbf{j}, \mathbf{k} \in \mathbf{N}^n},$$

where C_n is a constant which depends only on the dimension and the diagonal entries $\mu(|\mathbf{k}|, \lambda)$ can be expressed explicitly in terms a Laguerre transform. Denoting $\kappa = |\mathbf{k}|$ we in fact have

$$\mu(\kappa, \lambda) = c_\kappa^{n-1} \int_0^\infty K_0^\lambda(s)\Lambda_\kappa^{n-1}(\frac{1}{2}|\lambda|s^2)(\frac{1}{2}|\lambda|s^2)^{\frac{1-n}{2}}s^{2n-1}ds,$$

where $c_\kappa^\delta = (\frac{\kappa!}{(\kappa+\delta)!})^{1/2}$, $\Lambda_\kappa^\delta(x)$ is a Laguerre function³ of type δ , and $K^\lambda(z) = \int_{\mathbf{R}} K(z, t)e^{i\lambda t}dt$.

It therefore follows that the operators $M(\lambda)$ are bounded on $L^2(\mathbf{R}^n)$ if and only if the diagonal scalars $\mu(\kappa, \lambda)$ are bounded uniformly in κ , and hence

$$\|T^\epsilon f\|_{L^2(\mathbf{H}^n)} \leq A\|f\|_{L^2(\mathbf{H}^n)} \quad \text{if and only if} \quad |\mu(\kappa, \lambda)| \lesssim A \text{ uniformly in } \kappa \text{ and } \lambda \neq 0.$$

For more on the group Fourier transform and Laguerre functions see [9] and [11].

³ For the definition and properties of Laguerre functions see §3.2.

2.2. Main estimates. We have seen that matters therefore naturally reduce to the study of the ‘Fourier transforms’ $\mu(\kappa, \lambda)$. Making the change of variables

$$(s, t) \mapsto (r \sin^{\frac{1}{2}} \theta, r^2 \cos \theta) \quad \text{where } 0 < r < \infty \text{ and } 0 \leq \theta \leq \pi,$$

it follows that $s^4 + t^2 = r^4$, $ds dt = r^2 \sin^{-\frac{1}{2}} \theta dr d\theta$, and hence that

$$\mu(\kappa, \lambda) = c_\kappa^{n-1} \int_0^\pi \int_0^\infty (\sin \theta)^{n-1} \chi_\epsilon(r) r^{-1-\alpha} e^{i[r^{-\beta} + \lambda r^2 \cos \theta]} \left(\frac{1}{2} \lambda r^2 \sin \theta\right)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1} \left(\frac{1}{2} \lambda r^2 \sin \theta\right) dr d\theta,$$

where $\chi_\epsilon(r) = e^{-\epsilon r^{-\beta}} \chi(r)$.

It is natural to consider the cases for bounded and unbounded κ separately. We now fix κ_0 to be a large constant. Our main result for bounded κ is the following.

Theorem 2.1. *If $\kappa \leq \kappa_0$ and $|\lambda| \gg 1$ then we have the asymptotic expansion,*

$$(2) \quad \mu(\kappa, \lambda) = c_1 |\lambda|^{\frac{\alpha - (n + \frac{1}{2})\beta}{\beta + 2}} e^{ic_2 |\lambda|^{\frac{\beta}{\beta + 2}}} + O(|\lambda|^{\frac{\alpha - (n + 1 - \epsilon)\beta}{\beta + 2}}).$$

For unbounded κ we write $\mu(\kappa, \lambda) = \sum_{j=0}^\infty \mu_j(\kappa, \lambda)$, where

$$\mu_j(\kappa, \lambda) = c_\kappa^{n-1} \int_0^\pi \int_0^\infty (\sin \theta)^{n-1} \vartheta(2^j r) \chi_\epsilon(r) r^{-1-\alpha} e^{i[r^{-\beta} + \lambda r^2 \cos \theta]} x^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr d\theta,$$

with $x = \frac{1}{2} \lambda r^2 \sin \theta$ and $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ chosen such that $\sum_{j=0}^\infty \vartheta(2^j r) = 1$ for all $0 \leq r \leq 1$. Our main result for unbounded κ is then the following.

Theorem 2.2. *The following estimate holds uniformly for $\kappa \geq \kappa_0$ and $\lambda \neq 0$,*

$$|\mu_j(\kappa, \lambda)| \leq C \min\{2^{j(\alpha - n\beta)}, (1 + |\lambda|\kappa)^{\frac{\alpha - n\beta}{2(\beta + 1)}}\}.$$

Remark 2.3. Theorem 2.2 alone only implies $|\mu(\kappa, \lambda)| \leq C \log(1 + |\lambda|\kappa)(1 + |\lambda|\kappa)^{\frac{\alpha - n\beta}{2(\beta + 1)}}$ for $\kappa \geq \kappa_0$.

2.3. Proof of Theorem 1.1. Let us now see how Theorems 2.1 and 2.2 imply Theorem 1.1.

It is clear that part (ii) of Theorem 1.1 follows immediately from (2). Furthermore, one easily obtains the estimate $|\mu(\kappa, \lambda)| \leq C$ for all $0 < |\lambda| \leq \lambda_0$, for some fixed constant λ_0 . It follows from the Laguerre function estimates in §3.2 that this estimate holds uniformly for all κ , however Theorem 2.2 shows us that for fixed λ we in fact have decay in κ .

In order to prove Theorem 1.1 (i) we shall dyadically decomposing our operator and using almost orthogonality. Recall the following version of Cotlar’s Lemma,

Cotlar’s Lemma. *Suppose $\{T_i\}$ is a finite collection of bounded operators on L^2 . If in addition these operators satisfy, for some $\delta > 0$, the almost orthogonality condition*

$$\|T_i^* T_j\|_{Op} + \|T_i T_j^*\|_{Op} \leq C 2^{-\delta|i-j|},$$

then

$$\left\| \sum_i T_i \right\|_{Op} \leq A,$$

where the constant A is independent of the number of these T_j .

For an elegant proof of this result see [9]. If we now define the dyadic operator $T_j = f * K_j$, where

$$K_j(z, t) = \vartheta(2^j \rho(z, t)) K^\epsilon(z, t),$$

it then follows that

$$T^\epsilon f(z, t) = \sum_{j=0}^{\infty} T_j f(z, t).$$

Now the operators T_j are bounded on $L^2(\mathbf{H}^n)$ if and only if

$$|\mu_j(\kappa, \lambda)| \leq C,$$

uniformly in κ and $\lambda \neq 0$. It therefore an immediate consequence of Theorem 2.2 that the norms of T_j are uniformly bounded whenever $\alpha \leq n\beta$. Theorem 1.1 now follows from an application of Cotlar's lemma (and a standard limiting argument) once we have verified that our operators T_j are, in the following sense, almost orthogonal.

Lemma 2.4. *If $\alpha = n\beta$ then $\|T_i^* T_j\|_{Op} + \|T_i T_j^*\|_{Op} \leq C 2^{-n\beta|i-j|}$.*

Proof. This follows trivially whenever $|i - j| \leq 10$, since $\|T_i^* T_j\|_{Op} \leq \|T_i\|_{Op} \|T_j\|_{Op}$. We shall therefore, without loss of generality, assume that $j \geq i + 10$. Now $T_i^* T_j$ has a kernel $L_{ij}(z, t) = K_j * \bar{K}_i([z, t]^{-1})$, and the same operator norm as the operator with kernel the L^1 dilate of L_{ij} , namely

$$\begin{aligned} \tilde{L}_{ij}([z, t]^{-1}) &= 2^{-jQ} L_{ij}(2^{-j} \circ [z, t]^{-1}) \\ &= 2^{-jQ} \int K_j(w, s) \bar{K}_i((2^{-j} \circ [z, t]) \cdot [w, s]) dw ds \\ &= 2^{-j2Q} \int K_j(2^{-j}w, 2^{-2j}s) \bar{K}_i([2^{-j}z, 2^{-2j}t] \cdot [2^{-j}w, 2^{-2j}s]) dw ds \\ &= 2^{j2\alpha} \int_{\substack{\rho(w,s) \sim 1 \\ \rho([z,t] \cdot [w,s]) \sim 2^{j-i}}} \rho(w, s)^{-Q-\alpha} \rho([z, t] \cdot [w, s])^{-Q-\alpha} e^{i2^{j\beta}[\rho(w,s)^{-\beta} - \rho([z,t] \cdot [w,s])^{-\beta}]} dw ds. \end{aligned}$$

We trivially obtain the estimate $|\tilde{L}_{ij}([z, t]^{-1})| \leq C 2^{j2\alpha} 2^{(i-j)(Q+\alpha)}$, while integration by parts $2n$ times gives the favorable estimate

$$|\tilde{L}_{ij}([z, t]^{-1})| \leq C 2^{j(2\alpha-2n\beta)} 2^{(i-j)(Q+\alpha)} = 2^{(i-j)(Q+\alpha)}.$$

This of course implies that $\|\tilde{L}_{ij}\|_{L^1(\mathbf{H}^n)} \leq C 2^{(i-j)\alpha}$. \square

So we have reduced matters to proving Theorem 2.1 and Theorem 2.2, we present these arguments in Sections 4 and 5 respectively. As with the Euclidean case the asymptotics of special functions, in this case Laguerre polynomials, will be crucial. We discuss these expansions in Section 3 as well as stating some standard results for oscillatory integrals in one dimension that we shall require.

3. PRELIMINARIES

3.1. Oscillatory integrals. We present here a very brief overview of the theory of oscillatory integrals in one dimension, this gives an essentially complete description of the behavior of integrals

of the form

$$I(\lambda) = \int_a^b e^{i\lambda\Phi(x)}\Psi(x)dx, \quad \lambda > 0,$$

as $\lambda \rightarrow \infty$, where Φ and Ψ are smooth functions. The behavior of $I(\lambda)$ is governed by three basic principles: *localization*, *scaling*, and *asymptotics*. We shall present these respective principles as three propositions: the first of these can be thought of as a principle of non-stationary phase, the second is one of van der Corput's lemmas, and the third is a formulation of the method of stationary phase; for proofs see [9] or [13].

Proposition 3.1. *Suppose Ψ has compact support in (a, b) and Φ' is never vanishes, then for all $N \geq 0$ we have*

$$|I(\lambda)| \leq C_{N,\Phi,\Psi}\lambda^{-N}.$$

Proposition 3.2. *Suppose Φ is real-valued and $|\Phi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$, then*

$$|I(\lambda)| \leq C_k\lambda^{-\frac{1}{k}} \left[|\Psi(b)| + \int_a^b |\Psi'(x)|dx \right],$$

whenever (i) $k = 1$ and $\Phi''(x)$ has at most one zero, or (ii) $k \geq 2$.

Proposition 3.3. *Suppose Φ is real-valued, $\Phi'(x_0) = 0$, while $\Phi''(x_0) \neq 0$. If Ψ is supported in a sufficiently small neighborhood of x_0 , then*

$$I(\lambda) = e^{i\lambda\Phi(x_0)}\sigma(\lambda),$$

where σ is a symbol of order $-\frac{1}{2}$, that is $|\sigma^{(\ell)}(\lambda)| \leq c_\ell(1 + \lambda)^{-\frac{1}{2}-\ell}$. The constant c_ℓ depends on the $C^{\ell+1}$ norms of Φ and Ψ on the $\text{supp } \Psi$, the size of this support, and on a lower bound for $|\Phi''(x_0)|$.

Example 3.4. The *Bessel functions*, defined for real $k > -\frac{1}{2}$ by the formula

$$J_k(\lambda) = (\pi^{\frac{1}{2}}\Gamma(k + \frac{1}{2}))^{-1} \left(\frac{\lambda}{2}\right)^k \int_{-1}^1 e^{i\lambda t}(1 - t^2)^{k-\frac{1}{2}} dt$$

are a model case for these oscillatory integrals and using the Propositions above one can show that

$$(3) \quad J_k(\lambda) = \sigma_1(\lambda)e^{i\lambda} + \sigma_2(\lambda)e^{-i\lambda},$$

where σ_i is a symbol of order $-\frac{1}{2}$ for $i = 1, 2$.

3.2. Asymptotic properties of Laguerre functions. Recall that Laguerre functions of type δ , $\delta > -1$, form an orthonormal basis for $L^2(\mathbf{R}^+)$ and are given by

$$(4) \quad \Lambda_\kappa^\delta(x) = \left(\frac{\kappa!}{(\kappa+\delta)!}\right)^{1/2} L_\kappa^\delta(x) e^{-\frac{1}{2}x} x^{\frac{\delta}{2}},$$

where $L_\kappa^\delta(x) = \sum_{j=0}^{\kappa} \binom{\kappa+\delta}{\kappa-j} \frac{(-x)^j}{j!}$ are the Laguerre polynomials of type δ .

The two asymptotic formulae below which hold uniformly in their respective ranges of validity (which overlap) are due to Erdélyi [2]; see also [5]. In what follows $\nu = 4\kappa + 2\delta + 2$ and $N = \nu/4$.

3.2.1. The Bessel asymptotic forms. Let $0 \leq x \leq b\nu$, $b < 1$. Then for $\kappa \geq \kappa_0$,

$$\Lambda_\kappa^\delta(x) = \left(\frac{(\delta + \kappa)!}{\kappa!}\right)^{\frac{1}{2}} 2^{\delta - \frac{1}{2}} \nu^{-\frac{\delta}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \{J_\delta(\nu\psi) + O[\nu^{-1}(\frac{x}{\nu-x})^{\frac{1}{2}} \widetilde{J}_\delta(\nu\psi)]\},$$

and so

$$(5) \quad \Lambda_\kappa^\delta(x) = C_1(\delta) \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \{J_\delta(\nu\psi) + O[\nu^{-1}(\frac{x}{\nu-x})^{\frac{1}{2}} \widetilde{J}_\delta(\nu\psi)]\}$$

where $C_1(\delta)$ is a constant independent of κ , $\psi = \psi(t)$ satisfies

$$(6) \quad \psi'(t) = \frac{1}{2} \left(\frac{1}{t} - 1\right)^{\frac{1}{2}}$$

and $t = \frac{x}{\nu}$. For $0 \leq t < 1$,

$$\psi(t) = \frac{1}{2} [(t - t^2)^{\frac{1}{2}} + \sin^{-1} t^{\frac{1}{2}}],$$

and

$$\widetilde{J}_\delta(u) = \begin{cases} J_\delta(u) & \text{if } u \text{ sufficiently small,} \\ (|J_\delta(u)|^2 + |Y_\delta(u)|^2)^{\frac{1}{2}} & \text{otherwise,} \end{cases}$$

here Y_δ and J_δ are Bessel functions of order δ .

Lemma 3.5. *If $0 \leq t \leq \frac{1}{2}$, then $\frac{1}{2}t^{\frac{1}{2}} \leq \psi(t) \leq t^{\frac{1}{2}}$.*

Proof. Let $f(t) = (t - t^2)^{\frac{1}{2}} + \sin^{-1} t^{\frac{1}{2}}$, notice then that $f'(t) = (\frac{1-t}{t})^{\frac{1}{2}}$. Now if $0 \leq s \leq \frac{1}{2}$, we have $\frac{1}{2}s^{-\frac{1}{2}} \leq f'(s) \leq s^{-\frac{1}{2}}$, and so

$$\frac{1}{2} \int_0^t s^{-\frac{1}{2}} ds \leq \int_0^t f'(s) ds \leq \int_0^t s^{-\frac{1}{2}} ds$$

which implies $t^{\frac{1}{2}} \leq f(t) \leq 2t^{\frac{1}{2}}$, since $f(0) = 0$. □

3.2.2. The Airy asymptotic forms. Let $0 < a\nu \leq x$, $a > 0$. Then for $\kappa \geq \kappa_0$,

$$\Lambda_\kappa^\delta(x) = \frac{(-1)^\kappa}{(\kappa!(\delta + \kappa)!)^{\frac{1}{2}}} 2^{\frac{5}{6}} N^{N+\frac{1}{6}} e^{-N} x^{-\frac{1}{2}} \left(\frac{\pi}{-\phi'}\right)^{\frac{1}{2}} \{\text{Ai}(-\nu^{\frac{2}{3}}\phi) + O[x^{-1} \widetilde{\text{Ai}}(-\nu^{\frac{2}{3}}\phi)]\},$$

and so, using Stirling's formula

$$(7) \quad \Lambda_\kappa^\delta(x) = C_2(\delta) (-1)^\kappa \nu^{\frac{1}{6}} x^{-\frac{1}{2}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} \{\text{Ai}(-\nu^{\frac{2}{3}}\phi) + O[x^{-1} \widetilde{\text{Ai}}(-\nu^{\frac{2}{3}}\phi)]\}$$

where $C_2(\delta)$ is a constant independent of κ , $\phi = \phi(t)$ satisfies

$$(8) \quad [\phi(t)]^{\frac{1}{2}} \phi'(t) = \frac{1}{2} \left(\frac{1}{t} - 1\right)^{\frac{1}{2}},$$

and again $t = \frac{x}{\nu}$. Now one can show

$$\phi(t) = \left(\frac{3}{4}\right)^{\frac{2}{3}} \begin{cases} [\cos^{-1} t^{\frac{1}{2}} - (t - t^2)^{\frac{1}{2}}]^{\frac{2}{3}} & \text{if } 0 < t \leq 1, \\ -[(t - t^2)^{\frac{1}{2}} - \cosh^{-1} t^{\frac{1}{2}}]^{\frac{2}{3}} & \text{if } t > 1, \end{cases}$$

and

$$\widetilde{\text{Ai}}(z) = \begin{cases} \text{Ai}(z) & \text{if } z \geq 0, \\ (|\text{Ai}(z)|^2 + |\text{Bi}(z)|^2)^{\frac{1}{2}} & \text{if } z \leq 0, \end{cases}$$

here Ai and Bi are Airy integrals⁴.

Lemma 3.6. *If $\frac{1}{2} \leq t \leq 1$, then $\frac{1}{2}(1-t) \leq \phi(t) \leq 1-t$.*

Proof. Let $g(t) = \cos^{-1} t^{\frac{1}{2}} - (t-t^2)^{\frac{1}{2}}$, notice then that $g'(t) = -\left(\frac{1-t}{t}\right)^{\frac{1}{2}}$. Now if $\frac{1}{2} \leq s \leq 1$, we have $(1-s)^{\frac{1}{2}} \leq -g'(s) \leq 2(1-s)^{\frac{1}{2}}$, and so

$$\int_t^1 (1-s)^{\frac{1}{2}} ds \leq -\int_t^1 g'(s) ds \leq 2 \int_t^1 (1-s)^{\frac{1}{2}} ds$$

which implies $\frac{2}{3}(1-t)^{\frac{3}{2}} \leq g(t) \leq \frac{4}{3}(1-t)^{\frac{3}{2}}$, since $g(1) = 0$. □

Note also that, for $z > 0$

$$\begin{aligned} \text{Ai}(-z) &= \frac{1}{3}z^{\frac{1}{2}}[J_{1/3}(\frac{2}{3}z^{\frac{3}{2}}) + J_{-1/3}(\frac{2}{3}z^{\frac{3}{2}})] \\ \text{Bi}(-z) &= \left(\frac{z}{3}\right)^{\frac{1}{2}}[J_{1/3}(\frac{2}{3}z^{\frac{3}{2}}) + J_{-1/3}(\frac{2}{3}z^{\frac{3}{2}})]. \end{aligned}$$

3.2.3. Trivial Estimates. It follows from the asymptotics above that for κ large we have the following crude estimates for our Laguerre function; see Askey and Wainger [1].

$$|\Lambda_{\kappa}^{\delta}(x)| \leq C \begin{cases} (x\nu)^{\frac{\delta}{2}} & \text{if } 0 \leq x \leq \frac{1}{\nu}, \\ (x\nu)^{-\frac{1}{4}} & \text{if } \frac{1}{\nu} \leq x \leq \frac{\nu}{2}, \\ \nu^{-\frac{1}{4}}(\nu-x)^{-\frac{1}{4}} & \text{if } \frac{\nu}{2} \leq x \leq \nu - \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{3}} & \text{if } \nu - \nu^{\frac{1}{3}} \leq x \leq \nu + \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{4}}(x-\nu)^{-\frac{1}{4}}e^{-\gamma_1\nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}} & \text{if } \nu + \nu^{\frac{1}{3}} \leq x \leq \frac{3\nu}{2}, \\ e^{-\gamma_2 x} & \text{if } x \geq \frac{3\nu}{2}, \end{cases}$$

where $\gamma_1, \gamma_2 > 0$ are fixed constants.

4. BOUNDED DEGREE CASE: PROOF OF THEOREM 2.1

We shall now prove Theorem 2.1, and hence assume $\kappa \leq \kappa_0$ for some large constant κ_0 . To obtain estimates for $\mu(\kappa, \lambda)$ we may, with no loss in generality, assume that $\lambda > 0$. If we now perform the rescaling $r \mapsto \lambda^{-\frac{1}{\beta+2}}r$ we see that

$$\mu(\kappa, \lambda^{\beta+2}) = c_{\kappa}^{n-1} \lambda^{\alpha} \int_0^{\pi} \int_0^{\infty} (\sin \theta)^{n-1} \chi_{\epsilon}\left(\frac{r}{\lambda}\right) r^{-1-\alpha} e^{i\lambda^{\beta}\varphi(r,\theta)} \left(\frac{1}{2}\lambda^{\beta}r^2 \sin \theta\right)^{\frac{1-n}{2}} \Lambda_{\kappa}^{n-1}\left(\frac{1}{2}\lambda^{\beta}r^2 \sin \theta\right) dr d\theta,$$

where $\varphi(r, \theta) = r^{-\beta} + r^2 \cos \theta$.

It is easy to see that Theorem 2.1 is then equivalent to the following result.

Theorem 4.1. *If $\kappa \leq \kappa_0$ and $\lambda \gg 1$, then we have the asymptotic expansion*

$$\mu(\kappa, \lambda^{\beta+2}) = c_1 \lambda^{\alpha - (n + \frac{1}{2})\beta} e^{ic_2 \lambda^{\beta}} + O(\lambda^{\alpha - (n+1-\epsilon)\beta}).$$

⁴ Recall that Ai(z) and Bi(z) are independent solutions of the differential equation $\frac{d^2 y}{dz^2} = zy$ and have the integral representations $\text{Ai}(z) = \frac{1}{\pi} \int_0^{\infty} \cos(\frac{1}{3}t^3 + zt) dt$ and $\text{Bi}(z) = \frac{1}{\pi} \int_0^{\infty} \left\{ e^{\frac{1}{3}t^3 + zt} + \sin(\frac{1}{3}t^3 + zt) \right\} dt$.

Let us introduce some notation, it follow from (4) that for $\lambda > 0$

$$\mu(\kappa, \lambda^{\beta+2}) = \int_0^\pi (\sin \theta)^{n-1} \mathcal{I}(\theta) d\theta = \int_0^\infty \chi_\epsilon\left(\frac{r}{\lambda}\right) r^{-1-\alpha} e^{i\lambda^\beta r^{-\beta}} \mathcal{J}(r) dr,$$

where

$$\mathcal{I} = \mathcal{I}(\theta) = (c_\kappa^{n-1})^2 \lambda^\alpha \int_0^\infty \chi_\epsilon\left(\frac{r}{\lambda}\right) r^{-1-\alpha} e^{i\lambda^\beta \varphi(r, \theta)} L_\kappa^{n-1}\left(\frac{1}{2}\lambda^\beta r^2 \sin \theta\right) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} dr,$$

and

$$\mathcal{J} = \mathcal{J}(r) = (c_\kappa^{n-1})^2 \lambda^\alpha \int_0^\pi (\sin \theta)^{n-1} e^{i\lambda^\beta r^2 \cos \theta} L_\kappa^{n-1}\left(\frac{1}{2}\lambda^\beta r^2 \sin \theta\right) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} d\theta.$$

Proof of Theorem 4.1. Let C_β be a suitably small constant to be determined.

Region A: $0 \leq r \leq C_\beta$. Provided that C_β is chosen small enough we have that

$$-\partial_r \varphi(r, \theta) = \beta r^{-(\beta+1)} - 2r \cos \theta \geq C r^{-(\beta+1)}.$$

It then essentially follows from Proposition 3.1 (as $\chi_\epsilon\left(\frac{r}{\lambda}\right)$ and all of its derivatives vanish at $r = 0$) that

$$|\mathcal{I}(\theta)| \leq C \lambda^{\alpha-N\beta},$$

for all $N \geq 0$, we of course then obtain that $|\mu(\kappa, \lambda^{\beta+2})| \leq C \lambda^{\alpha-N\beta}$.

Region B: $C_\beta \leq r \leq \lambda$. We now certainly have that $\mu(\kappa, \lambda)$ is absolutely convergent, but now the phase in \mathcal{I} may be critical.

Case 1: If $\lambda^{-\beta(1-\epsilon)} \leq r^2 \sin \theta \leq r^2$, then $\lambda^\beta r^2 \sin \theta \geq \lambda^{\beta\epsilon}$. It follows immediately that

$$L_\kappa^{n-1}\left(\frac{1}{2}\lambda^\beta r^2 \sin \theta\right) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} \leq C \lambda^{-N\beta},$$

for all $N \geq 0$ and hence that $|\mu(\kappa, \lambda^{\beta+2})| \leq C \lambda^{\alpha-N\beta}$.

Case 2: If $0 \leq r^2 \sin \theta \leq \lambda^{-\beta(1-\epsilon)}$, then it follows that either $\cos \theta \approx 1$ or $\cos \theta \approx -1$.

(i) We first note that if $\frac{\pi}{2} \leq \theta \leq \pi$ then $\cos \theta \approx -1$ and it follows that

$$-\partial_r \varphi(r, \theta) = \beta r^{-(\beta+1)} - 2r \cos \theta \geq C_\beta,$$

and that we can argue as in *Region A*.

(ii) Now if $0 \leq \theta \leq \frac{\pi}{2}$ then $\cos \theta \approx 1$ and we shall write $\mathcal{J}(r) = \mathcal{M}(r) + \mathcal{E}(r)$, where

$$\mathcal{M}(r) = (c_\kappa^{n-1})^2 \lambda^\alpha e^{i\lambda^\beta r^2} \int_0^{\frac{\pi}{2}} (\sin \theta)^{n-1} L_\kappa^{n-1}\left(\frac{1}{2}\lambda^\beta r^2 \sin \theta\right) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} d\theta,$$

$$\mathcal{E}(r) = (c_\kappa^{n-1})^2 \lambda^\alpha \int_0^{\frac{\pi}{2}} (\sin \theta)^{n-1} L_\kappa^{n-1}\left(\frac{1}{2}\lambda^\beta r^2 \sin \theta\right) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} [e^{i\lambda^\beta r^2 \cos \theta} - e^{i\lambda^\beta r^2}] d\theta.$$

It is then easy to see that

$$|\mathcal{E}(r)| \leq C \lambda^{\alpha+\beta} r^2 \int_{0 \leq r^2 \sin \theta \leq \lambda^{-\beta(1-\epsilon)}} (\sin \theta)^{n+1} d\theta \leq C \lambda^{\alpha-(n+1-\epsilon')\beta} r^{-2n-2}.$$

Now for the main term we write $\mathcal{M}(r) = \mathcal{M}_1(r) + \mathcal{M}_2(r)$, where

$$\mathcal{M}_1(r) = (c_\kappa^{n-1})^2 \lambda^\alpha e^{i\lambda^\beta r^2} \int_0^{\frac{\pi}{2}} (\sin \theta)^{n-1} L_\kappa^{n-1}(\frac{1}{2}\lambda^\beta r^2 \sin \theta) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} \cos \theta d\theta,$$

$$\mathcal{M}_2(r) = (c_\kappa^{n-1})^2 \lambda^\alpha \int_0^{\frac{\pi}{2}} (\sin \theta)^{n-1} L_\kappa^{n-1}(\frac{1}{2}\lambda^\beta r^2 \sin \theta) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} [1 - \cos \theta] d\theta.$$

In a similar manner to the estimate for $\mathcal{E}(r)$ above we obtain the estimate

$$|\mathcal{M}_2(r)| \leq C \lambda^\alpha \int_{0 \leq r^2 \sin \theta \leq \lambda^{-\beta(1-\varepsilon)}} (\sin \theta)^{n+1} d\theta \leq C \lambda^{\alpha - (n+2-\varepsilon)\beta} r^{-2n-4}.$$

Now for the main term $\mathcal{M}_1(r)$ we have

$$\begin{aligned} \mathcal{M}_1(r) &= (c_\kappa^{n-1})^2 \lambda^\alpha e^{i\lambda^\beta r^2} \int_{0 \leq r^2 \sin \theta \leq \lambda^{-\beta(1-\varepsilon)}} (\sin \theta)^{n-1} L_\kappa^{n-1}(\frac{1}{2}\lambda^\beta r^2 \sin \theta) e^{-\frac{1}{4}\lambda^\beta r^2 \sin \theta} \cos \theta d\theta \\ &= (c_\kappa^{n-1})^2 \lambda^\alpha e^{i\lambda^\beta r^2} \lambda^{-n\beta} r^{-2n} \int_0^{\lambda^{\beta\varepsilon}} u^{n-1} L_\kappa^{n-1}(\frac{1}{2}u) e^{-\frac{1}{4}u} du \\ &= C \lambda^{\alpha-n\beta} e^{i\lambda^\beta r^2} r^{-2n} + O(\lambda^{\alpha-N\beta}). \end{aligned}$$

We have therefore shown that

$$\mu(\kappa, \lambda^{\beta+2}) = C \lambda^{\alpha-n\beta} M + O(\lambda^{\alpha-(n+1-\varepsilon)\beta}),$$

where

$$M = \int_0^\infty \chi_\varepsilon\left(\frac{r}{\lambda}\right) r^{-2n-1-\alpha} e^{i\lambda^\beta \varphi(r,0)} dr.$$

Now notice that

$$\partial_r^2 \varphi(r, 0) = \beta(\beta+1)r^{-(\beta+2)} + 2 \geq 2,$$

it follows that $\partial_r \varphi(r, 0)$ can vanish at most one point $r = r_0$. Suppose that we are in a suitably small neighborhood of r_0 , outside of this interval we have that $|\partial_r \varphi(r, 0)| \geq C > 0$ and can argue as in *Region A*. Note that for all $\ell = 0, 1, \dots$

$$|\partial_r^\ell \varphi(r, 0)| + |\partial_r^\ell [\chi_\varepsilon\left(\frac{r}{\lambda}\right) r^{-2n-1-\alpha}]| \leq c_\ell.$$

We may therefore, in a suitably small neighborhood of r_0 , apply the method of stationary phase (Proposition 3.3) and obtain that

$$M = C(r_0) \lambda^{-\frac{1}{2}\beta} e^{i\lambda^\beta \varphi(r_0,0)} + O(\lambda^{-\frac{3}{2}\beta}). \quad \square$$

5. UNBOUNDED DEGREE CASE: PROOF OF THEOREM 2.2

We shall now prove Theorem 2.2, and hence assume $\kappa \geq \kappa_0$, for some large constant κ_0 . We may, as in Section 4, assume $\lambda > 0$. Performing the rescaling $r \mapsto 2^{-j}r$ we see that

$$\mu_j(\kappa, \lambda 2^{2j}) = c_\kappa^{n-1} 2^{j\alpha} \int_0^\pi \int_0^\infty (\sin \theta)^{n-1} \vartheta(r) r^{-1-\alpha} e^{i2^{j\beta} \varphi(r,\theta)} x^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr d\theta,$$

where now

$$\varphi(r, \theta) = r^{-\beta} + \lambda 2^{-j\beta} r^2 \cos \theta \quad \text{and} \quad x = x(\lambda, r, \theta) = \frac{1}{2} \lambda r^2 \sin \theta.$$

It is then easy to see that Theorem 2.2 will be a consequence of the following result.

Theorem 5.1. *If $\kappa \geq \kappa_0$ then following estimate holds uniformly in κ and $\lambda \neq 0$,*

$$|\mu_j(\kappa, \lambda 2^{2j})| \leq C 2^{j\alpha} \min\{2^{-jn\beta}, (1 + \lambda\kappa)^{-\frac{n}{2}}\}.$$

In order to estimate $\mu_j(\kappa, \lambda 2^{2j})$ we shall make use of the asymptotics for Laguerre functions presented in §3.2, it is then natural to consider six separate regions and write

$$\mu_j(\kappa, \lambda 2^{2j}) = \mu_{j,1}(\kappa, \lambda 2^{2j}) + \cdots + \mu_{j,6}(\kappa, \lambda 2^{2j})$$

where for $k = 1, \dots, 6$,

$$\mu_{j,k}(\kappa, \lambda 2^{2j}) = c_\kappa^{n-1} 2^{j\alpha} \int_0^\pi \int_0^\infty \eta_k(x) (\sin \theta)^{n-1} \vartheta(r) r^{-1-\alpha} e^{i2^{j\beta} \varphi(r, \theta)} x^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr d\theta,$$

and each $\eta_k(x)$ smoothly localizes x to the k th interval indicated in 3.2.3.

As in Section 4 we write, for $\lambda > 0$

$$\mu_{j,k}(\kappa, \lambda 2^{2j}) = \int_0^\pi (\sin \theta)^{n-1} \mathcal{I}_k(j, \theta) d\theta = \int_0^\infty \vartheta(r) r^{-1-\alpha} e^{i2^{j\beta} r^{-\beta}} \mathcal{J}_k(j, r) dr,$$

where, as we are now considering the case when $\kappa \geq \kappa_0$, we essentially have

$$\begin{aligned} \mathcal{I}_k &= \mathcal{I}_k(j, \theta) = 2^{j\alpha} \int_0^\infty \eta_k(x) \vartheta(r) r^{-1-\alpha} e^{i2^{j\beta} \varphi(r, \theta)} (x\nu)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr, \\ \mathcal{J}_k &= \mathcal{J}_k(j, r) = 2^{j\alpha} \int_0^\pi \eta_k(x) (\sin \theta)^{n-1} e^{i\lambda r^2 \cos \theta} (x\nu)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) d\theta. \end{aligned}$$

It will again be natural to consider separately the regions where we can and cannot integrate by parts in r . It is clear that we may choose a constant C_β small enough so that

$$-\partial_r \varphi(r, \theta) \geq C r^{-(\beta+1)} \quad \text{and} \quad |\partial_r^\ell \varphi(r, \theta)| \leq C r^{-(\beta+\ell)},$$

whenever $0 < \lambda 2^{-j\beta} \leq C_\beta$. However if we are to integrate by parts in r we must take care of what happens when the derivative hits the amplitude of \mathcal{I}_k .

Recall that $\Lambda_\kappa^\delta(x) = c_\kappa^\delta L_\kappa^\delta(x) e^{-\frac{1}{2}x} x^{\frac{\delta}{2}}$. Now since $\frac{d}{dx} L_\kappa^\delta(x) = -L_{\kappa-1}^{\delta+1}(x)$, and $c_{\kappa-1}^{\delta+1} = \kappa^{-\frac{1}{2}} c_\kappa^\delta$ it follows that

$$\frac{d}{dx} \Lambda_\kappa^\delta(x) = \frac{1}{2} \left(\frac{\delta}{x} - 1 \right) \Lambda_\kappa^\delta(x) - \left(\frac{\kappa}{x} \right)^{\frac{1}{2}} \Lambda_{\kappa-1}^{\delta+1}(x).$$

Therefore, using the fact that $\partial_r x = \frac{2x}{r}$ we see that

$$\partial_r \Lambda_\kappa^\delta(x) = -r^{-1} [(x - \delta) \Lambda_\kappa^\delta(x) + 2(x\kappa)^{\frac{1}{2}} \Lambda_{\kappa-1}^{\delta+1}(x)].$$

If we instead take N derivatives it is easy to see that we get

$$\partial_r^N \Lambda_\kappa^\delta(x) = r^{-N} [P_N(x) \Lambda_\kappa^\delta(x) + (x\kappa)^{\frac{1}{2}} P_{N-1}(x) \Lambda_{\kappa-1}^{\delta+1}(x) + \cdots + (x\kappa)^{\frac{N}{2}} P_0(x) \Lambda_{\kappa-N}^{\delta+N}(x)],$$

that is

$$(9) \quad \partial_r^N \Lambda_\kappa^\delta(x) = r^{-N} \sum_{\ell=0}^N (x\kappa)^{\frac{\ell}{2}} P_{N-\ell}(x) \Lambda_{\kappa-\ell}^{\delta+\ell}(x),$$

where $P_{N-\ell}(x)$ is some polynomial of degree $N - \ell$ in x . It is therefore only advantageous to integration by parts in r when both x and $(x\nu)^{\frac{1}{2}}$ are dominated by $2^{j\beta}$ and hence we shall in general only do so when

$$\max\{(\lambda\nu)^{\frac{1}{2}}, \lambda\} \leq C_\beta 2^{j\beta}.$$

In actual fact we shall establish, for each k , the following

$$|\mu_{j,k}(\kappa, \lambda 2^{2j})| \leq C 2^{j\alpha} \begin{cases} 2^{-jn\beta} & \text{if } \max\{(\lambda\nu)^{\frac{1}{2}}, \lambda\} \leq C_\beta 2^{j\beta}, \\ \min\{(\lambda\nu)^{-\frac{n}{2}}, \lambda^{-n}\} & \text{if } \max\{(\lambda\nu)^{\frac{1}{2}}, \lambda\} \geq C_\beta 2^{j\beta}, \end{cases}$$

and from this one easily establishes Theorem 5.1.

5.1. Neighborhood of 0: $0 \leq x \leq \frac{1}{\nu}$. Notice that here $x\nu \leq 1$ and $|\Lambda_\kappa^\delta(x)| \leq C(x\nu)^{\frac{\delta}{2}}$.

Region A: $\max\{(\lambda\nu)^{\frac{1}{2}}, \lambda\} \leq C_\beta 2^{j\beta}$. In this region we can integrate by parts. Using our trivial estimate it is easy to see that

$$|\partial_r^N \Lambda_\kappa^\delta(x)| \leq C(x\nu)^{\frac{\delta}{2}+N}.$$

Therefore integrating by parts N times we obtain the estimate

$$\begin{aligned} \int_0^\pi (\sin \theta)^{n-1} |\mathcal{I}_1(\theta)| d\theta &= 2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \left| \int_0^\infty \eta_1(x) \vartheta(r) r^{-1-\alpha} e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr \right| d\theta \\ &\leq C 2^{j(\alpha-N\beta)} \int_{\sin \theta \leq \min\{1, (\lambda\nu)^{-1}\}} (\sin \theta)^{n-1} d\theta \\ &\leq C 2^{j(\alpha-N\beta)} (1 + \lambda\nu)^{-n}, \end{aligned}$$

for any $N \geq 0$. So we obtain the estimate

$$|\mu_{j,1}(\kappa, \lambda 2^{2j})| \leq C 2^{j(\alpha-N\beta)} (1 + \lambda\nu)^{-n}.$$

Region B: $\max\{(\lambda\nu)^{\frac{1}{2}}, \lambda\} \geq C_\beta 2^{j\beta}$. Here we necessarily have $\sin \theta \leq 2(\lambda\nu)^{-1} \leq 2^{-j\beta}$ and therefore either $\cos \theta \approx 1$ or $\cos \theta \approx -1$.

(i) We first note that if $\frac{\pi}{2} \leq \theta \leq \pi$ then $\cos \theta \approx -1$ and it follows that

$$-\partial_r \varphi(r, \theta) = \beta r^{-(\beta+1)} - 2\lambda 2^{-j\beta} r \cos \theta \geq C(1 + \lambda 2^{-j\beta}),$$

we can therefore argue as in *Region A* and obtain the estimate

$$|\mu_{j,1}(\kappa, \lambda 2^{2j})| \leq C 2^{j\alpha} (\lambda\nu)^{-n} \min\{2^{-jN\beta}, \lambda^{-N}\}.$$

(ii) Now if $0 \leq \theta \leq \frac{\pi}{2}$ then $\cos \theta \approx 1$ and it is easy to see that

$$\partial_r^2 \varphi(r, \theta) = \beta r^{-(\beta+1)} + 2\lambda 2^{-j\beta} \cos \theta \geq C(1 + \lambda 2^{-j\beta}),$$

and hence by van der Corput's lemma we have $|\mathcal{I}_1(\theta)| \leq C 2^{j\alpha} \min\{2^{-j\frac{1}{2}\beta}, \lambda^{-\frac{1}{2}}\}$. It therefore follows that

$$\begin{aligned} |\mu_{j,1}(\kappa, \lambda 2^{2j})| &\leq C \int_0^\pi (\sin \theta)^{n-1} |\mathcal{I}_1(\theta)| d\theta \\ &\leq C 2^{j\alpha} \min\{2^{-j\frac{1}{2}\beta}, \lambda^{-\frac{1}{2}}\} \int_{\sin \theta \leq (\lambda\nu)^{-1}} (\sin \theta)^{n-1} d\theta \\ &\leq C 2^{j\alpha} (\lambda\nu)^{-n} \min\{2^{-j\frac{1}{2}\beta}, \lambda^{-\frac{1}{2}}\}. \end{aligned}$$

5.2. Oscillatory interval I: $\frac{1}{\nu} \leq x \leq \frac{\nu}{2}$. Notice that here $\frac{\nu}{2} \leq \nu - x < \nu$, and $|\Lambda_\kappa^\delta(x)| \leq C(x\nu)^{-\frac{1}{4}}$.

In this interval we shall make explicit use of the oscillation in the main term of our asymptotic expansion, which are in this case is given in terms of Bessel functions; see §3.2.1. We note here that from (6) and Lemma 3.5 it follows that

$$\psi' \sim \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \quad \text{while} \quad \psi \sim \left(\frac{x}{\nu}\right)^{\frac{1}{2}}.$$

The following estimates are then immediate,

$$\begin{aligned} (\nu\psi)^{-\frac{1}{2}} &\sim (x\nu)^{-\frac{1}{4}} \quad \text{and} \quad \partial_r[(\nu\psi)^{-\frac{1}{2}}] = -\frac{x}{r\nu^{\frac{3}{2}}}\psi^{-\frac{3}{2}}\psi' \sim -(x\nu)^{-\frac{1}{4}}, \\ \partial_r\nu\psi &= \frac{1}{r}x^{\frac{1}{2}}(\nu-x)^{\frac{1}{2}} \sim (x\nu)^{\frac{1}{2}} \quad \text{and} \quad \partial_r^2\nu\psi = -\frac{1}{r^2}\frac{x^{\frac{3}{2}}}{(\nu-x)^{\frac{1}{2}}} \sim x^{\frac{3}{2}}\nu^{-\frac{1}{2}}, \\ \left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} &\leq C \quad \text{and} \quad \left|\partial_r\left[\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}\right]\right| \leq C. \end{aligned}$$

Region A: $\max\{(\lambda\nu)^{\frac{1}{2}}, \lambda\} \leq C_\beta 2^{j\beta}$. In this region we shall again integrate by parts in r and since we have $x^N \leq (x\nu)^{\frac{N}{2}}$ it suffices to consider the integral

$$I_2(\theta) = 2^{j(\alpha-N\beta)} \int_0^\infty \eta_2(x)\vartheta(r)e^{i2^{j\beta}\varphi(r,\theta)}(x\nu)^{\frac{N+1-n}{2}}\Lambda_{\kappa-N}^{n-1+N}(x)dr.$$

Using the Bessel asymptotic forms (5) we may write $I_2 = cB + E_B$, where

$$B(\theta) = 2^{j(\alpha-N\beta)} \int_0^\infty \eta_2(x)\vartheta(r)e^{i2^{j\beta}\varphi(r,\theta)}(x\nu)^{\frac{N+1-n}{2}}\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}J_{n-1+N}(\nu\psi)dr.$$

ERROR TERM:

$$\begin{aligned} |E_B(\theta)| &\leq C2^{j(\alpha-N\beta)} \int_0^\infty \eta_2(x)\vartheta(r)(x\nu)^{\frac{N+1-n}{2}}\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}\nu^{-1}\left(\frac{x}{\nu-x}\right)^{\frac{1}{2}}|\tilde{J}_{n-1+N}(\nu\psi)|dr \\ &\leq C2^{j(\alpha-N\beta)}\nu^{-1} \int_{x \leq \nu} (x\nu)^{\frac{1}{2}(N-n-\frac{1}{2})}dx \\ &\leq C2^{j(\alpha-(n+\frac{1}{2})\beta)}, \end{aligned}$$

provided $N > n$, since $(x\nu)^{\frac{1}{2}} \leq C2^{j\beta}$.

MAIN TERM: Since $\nu\psi \geq (x\nu)^{\frac{1}{2}} \geq 1$ we shall utilize expression (3) and write $B = B_1 + B_2$, where

$$\begin{aligned} B_1(\theta) &= 2^{j(\alpha-N\beta)} \int_0^\infty \eta_2(x)\vartheta(r)e^{i2^{j\beta}[\varphi(r,\theta)+2^{-j\beta}\nu\psi]}(x\nu)^{\frac{N+1-n}{2}}\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}\sigma_1(\nu\psi)dr, \\ B_2(\theta) &= 2^{j(\alpha-N\beta)} \int_0^\infty \eta_2(x)\vartheta(r)e^{i2^{j\beta}[\varphi(r,\theta)-2^{-j\beta}\nu\psi]}(x\nu)^{\frac{N+1-n}{2}}\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}\sigma_2(\nu\psi)dr, \end{aligned}$$

where each σ_i is a symbol of order $-\frac{1}{2}$. Since $\partial_r\psi \sim \frac{1}{r}\psi$ it follows that

$$|\partial_r^\ell\sigma_i(\nu\psi)| \leq Cr^{-\ell}(\nu\psi)^{-\frac{1}{2}}.$$

Let us first consider the integral B_1 , and let $\Phi_\theta(r) = \varphi(r, \theta) + 2^{-j\beta}\nu\psi$. Now

$$\partial_r\Phi_\theta(r) = -\beta r^{-(\beta+1)} + \lambda 2^{-j\beta} 2r \cos \theta + 2^{-j\beta} \frac{1}{r} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}},$$

and so if C_β is chosen small enough it follows that

$$|\partial_r \Phi_\theta(r)| \geq C > 0.$$

In addition to this we also have

$$\partial_r^2 \Phi_\theta(r) = \beta(\beta+1)r^{-(\beta+2)} + \lambda 2^{-j\beta} 2 \cos \theta - 2^{-j\beta} \frac{1}{r^2} \frac{x^{\frac{3}{2}}}{(\nu-x)^{\frac{1}{2}}} \geq C > 0.$$

If we let

$$\Psi_\theta(r) = \vartheta(r)(x\nu)^{\frac{N+1-n}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \sigma_1(\nu\psi),$$

then for all $\ell = 0, 1, \dots$

$$|\partial_r^\ell \Psi_\theta(r)| \leq C(x\nu)^{\frac{N}{2}} (\nu\psi)^{-\frac{1}{2}}.$$

Applying van der Corput's lemma (integration by parts) gives

$$|B_1(\theta)| \leq C 2^{j(\alpha-(N+1)\beta)} \int_0^\infty \vartheta(r)(x\nu)^{\frac{2N+1-2n}{4}} dr \leq C 2^{j(\alpha-(n+\frac{1}{2})\beta)},$$

again provided $N > n$. Of course the phase in B_2 is never stationary, so we obtain the same estimate for B_2 , in *Region A* we therefore obtain

$$|\mu_{j,2}(\kappa, \lambda 2^{2j})| \leq C 2^{j(\alpha-(n+\frac{1}{2})\beta)}.$$

Region B: $\max\{(\lambda\nu)^{\frac{1}{2}}, \lambda\} \geq C_\beta 2^{j\beta}$. Here we shall not integrate by parts first and so wish to estimate

$$\mathcal{J}_2 = 2^{j\alpha} \int_0^\pi \eta_2(x)(\sin \theta)^{n-1} e^{i\lambda r^2 \cos \theta} (x\nu)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) d\theta.$$

Using the asymptotic forms for our Laguerre functions we can write $\mathcal{J}_2 = c\mathcal{B} + \mathcal{E}_B$, where

$$\mathcal{B}(r) = 2^{j\alpha} \int_0^\pi \eta_2(x)(\sin \theta)^{n-1} e^{i\lambda r^2 \cos \theta} (x\nu)^{\frac{1-n}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} J_{n-1}(\nu\psi) d\theta.$$

ERROR TERM:

$$\begin{aligned} |\mathcal{E}_B(r)| &\leq C 2^{j\alpha} \int_0^\pi \eta_2(x)(\sin \theta)^{n-1} (x\nu)^{\frac{1-n}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \nu^{-1} \left(\frac{x}{\nu-x}\right)^{\frac{1}{2}} |\tilde{J}_{n-1}(\nu\psi)| d\theta \\ &\leq C 2^{j\alpha} \int_{\sin \theta \leq \min\{1, \nu\lambda^{-1}\}} (\sin \theta)^{n-1} (x\nu)^{-\frac{1}{2}(n+\frac{1}{2})} d\theta \\ &\leq C 2^{j\alpha} \min\{(\lambda\nu)^{-\frac{1}{2}(n+\frac{1}{2})}, \lambda^{-n} \nu^{-\frac{1}{2}}\}. \end{aligned}$$

MAIN TERM : Since $\nu\psi \geq (x\nu)^{\frac{1}{2}} \geq 1$ we shall, as before, write $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, where

$$\mathcal{B}_1(r) = 2^{j\alpha} \int_0^\pi \eta_2(x)(\sin \theta)^{n-1} e^{i[\lambda r^2 \cos \theta + \nu\psi]} (x\nu)^{\frac{1-n}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \sigma_1(\nu\psi) d\theta,$$

$$\mathcal{B}_2(r) = 2^{j\alpha} \int_0^\pi \eta_2(x)(\sin \theta)^{n-1} e^{i[\lambda r^2 \cos \theta - \nu\psi]} (x\nu)^{\frac{1-n}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \sigma_2(\nu\psi) d\theta.$$

Let us first consider the integral \mathcal{B}_1 , if we let $\Phi_r(\theta) = \lambda r^2 \cos \theta + \nu\psi$ then we have

$$\begin{aligned} \partial_\theta \Phi_r(\theta) &= -\lambda r^2 \sin \theta + \nu \left(\frac{\nu-x}{x}\right)^{\frac{1}{2}} \frac{\lambda r^2 \cos \theta}{2\nu} \\ &= -\frac{1}{2} \lambda r^2 \left(2 \sin \theta - \left(\frac{\nu-x}{x}\right)^{\frac{1}{2}} \cos \theta\right). \end{aligned}$$

We shall now write $\mathcal{B}_1 = \mathcal{B}_1^1 + \mathcal{B}_1^2$, where

$$\mathcal{B}_1^1(r) = 2^{j\alpha} \int_0^\pi \chi(\sin \theta) \Psi_r(\theta) e^{i\Phi_r(\theta)} d\theta,$$

$$\mathcal{B}_1^2(r) = 2^{j\alpha} \int_0^\pi (1 - \chi(\sin \theta)) \Psi_r(\theta) e^{i\Phi_r(\theta)} d\theta,$$

and

$$\Psi_r(\theta) = (\sin \theta)^{n-1} (x\nu)^{\frac{1-n}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \sigma_1(\nu\psi).$$

Case 1: If $0 \leq \sin \theta \leq \frac{1}{10}$, then there are two possibilities; $\cos \theta \approx 1$ and $\cos \theta \approx -1$.

(i) If $0 \leq \theta \leq \frac{\pi}{2}$ then $\cos \theta \approx 1$ and

$$\left(\frac{\nu - x}{x}\right)^{\frac{1}{2}} \cos \theta \geq \frac{1}{2},$$

it therefore follows that

$$|\partial_\theta \Phi_r(\theta)| \geq (\sin \theta)^{-1} (x\nu)^{\frac{1}{2}}.$$

(ii) If $\frac{\pi}{2} \leq \theta \leq \pi$ then $\cos \theta \approx -1$ and we trivially have the same estimate as above.

Notice also that

$$|\partial_\theta^2 \Phi_r(\theta)| \leq C(\sin \theta)^{-2} (x\nu)^{\frac{1}{2}},$$

and since for all $\ell = 0, 1, \dots$ we have that $|\partial_\theta^\ell \sigma_i(\nu\psi)| \leq c_\ell (\sin \theta)^{-\ell} (x\nu)^{-\frac{1}{4}}$ it follows that if we let $\Psi_r^1(\theta) = \chi(\sin \theta) \Psi_r(\theta)$ then we have

$$|\Psi_r^1(\theta)| \leq (\sin \theta)^{n-1} (x\nu)^{\frac{1-n}{2} - \frac{1}{4}},$$

$$|\partial_\theta \Psi_r^1(\theta)| \leq (\sin \theta)^{n-2} (x\nu)^{\frac{1-n}{2} - \frac{1}{4}}.$$

An application of integration by parts therefore gives

$$\begin{aligned} |\mathcal{B}_1^1(r)| &\leq C 2^{j\alpha} \int_0^\pi \eta_2(x) |\partial_\theta ([\partial_\theta \Phi_r(\theta)]^{-1} \Psi_r^1(\theta))| d\theta \\ &\leq C 2^{j\alpha} \int_{\sin \theta \leq \min\{1, \nu\lambda^{-1}\}} (\sin \theta)^{n-1} (x\nu)^{\frac{1-n}{2} - \frac{3}{4}} d\theta \\ &\leq C 2^{j\alpha} \min\{(\lambda\nu)^{-\frac{1}{2}(n+\frac{1}{2})}, \lambda^{-n} \nu^{-\frac{1}{2}}\}. \end{aligned}$$

Case 2: If $\frac{1}{10} \leq \sin \theta \leq 1$, then $\lambda \leq 10\nu$ and $\Phi_r(\theta)$ may now be stationary, however we do have that

$$|\partial_\theta^2 \Phi_r(\theta)| \geq C(x\nu)^{\frac{1}{2}} \geq C(\lambda\nu)^{\frac{1}{2}},$$

and if we now let $\Psi_r^2(\theta) = \chi(\sin \theta) \Psi_r(\theta)$ it follows that

$$|\Psi_r^2(\theta)| + |\partial_\theta \Psi_r^2(\theta)| \leq C(x\nu)^{-\frac{1}{2}(n-\frac{1}{2})} \leq (\lambda\nu)^{-\frac{1}{2}(n-\frac{1}{2})}.$$

It therefore follows from an application of van der Corput's lemma that

$$|\mathcal{B}_1^2(r)| \leq C 2^{j\alpha} (\lambda\nu)^{-\frac{1}{4}} (\lambda\nu)^{-\frac{1}{2}(n-\frac{1}{2})} \leq C(\lambda\nu)^{-\frac{n}{2}}.$$

Bringing these two cases together we see that

$$|\mathcal{B}_1(r)| \leq C 2^{j\alpha} \min\{(\lambda\nu)^{-\frac{n}{2}}, \lambda^{-n} \nu^{-\frac{1}{2}}\}.$$

The integral \mathcal{B}_2 can be treated in exactly the same manner as above, in *Region B* we therefore obtain

$$|\mu_{j,2}(\kappa, \lambda 2^{2j})| \leq C 2^{j\alpha} \min\{(\lambda\nu)^{-\frac{n}{2}}, \lambda^{-n}\nu^{-\frac{1}{2}}\}.$$

5.3. Oscillatory interval II: $\frac{\nu}{2} \leq x \leq \nu - \nu^{\frac{1}{3}}$. We now have $\nu^{\frac{1}{3}} \leq \nu - x \leq \frac{\nu}{2}$ and $|\Lambda_\kappa^\delta(x)| \leq Cx^{-\frac{1}{4}}(\nu - x)^{-\frac{1}{4}}$. Notice also that $(\lambda\nu)^{\frac{1}{2}} \leq 2\lambda$.

The situation here is much the same as it was in §5.2 only here we must use instead the Airy asymptotic form. In order to do better than the trivial estimate we shall again make use the oscillation in the main term of our asymptotic expansion. It follows from Lemma 3.6 that

$$\phi \sim \frac{\nu - x}{\nu},$$

from this and (8) it is immediately clear that

$$\frac{1}{10} \leq \phi' \leq 10 \quad \text{and} \quad \phi'' \leq C.$$

Region A: $\lambda \leq C_\beta 2^{j\beta}$. In this region we should integrate by parts and since $x \leq (x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$I_3(\theta) = 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{N+1-n}{2}} \Lambda_{\kappa-N}^{n-1+N}(x) dr.$$

Using the Airy asymptotic forms (7) we may write this as $I_3 = cA + E_A$, where

$$A(\theta) = 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{N+1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} \text{Ai}(-\nu^{\frac{2}{3}}\phi) dr.$$

ERROR TERM:

$$\begin{aligned} |E_A(\theta)| &\leq C 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) (x\nu)^{\frac{N+1-n}{2}} x^{-\frac{3}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} |\widetilde{\text{Ai}}(-\nu^{\frac{2}{3}}\phi)| dr \\ &\leq C 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) (x\nu)^{\frac{1}{2}(N-\frac{1}{3}-n)} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} (\nu\phi^{\frac{3}{2}})^{-\frac{1}{2}} dr \\ &\leq C 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) (x\nu)^{\frac{1}{2}(N-\frac{1}{2}-n)} \nu^{\frac{1}{4}} (\nu - x)^{-\frac{1}{4}} dr \\ &\leq C 2^{j(\alpha-(n+\frac{1}{2})\beta)} \nu^{-\frac{3}{4}} \int_{\nu-x \leq \nu} (\nu - x)^{-\frac{1}{4}} dx \\ &\leq C 2^{j(\alpha-(n+\frac{1}{2})\beta)}, \end{aligned}$$

provided $N > n$.

MAIN TERM: Recall that for $z > 0$

$$\text{Ai}(-z) = \frac{1}{3} z^{\frac{1}{2}} [J_{1/3}(\frac{2}{3} z^{\frac{3}{2}}) + J_{-1/3}(\frac{2}{3} z^{\frac{3}{2}})],$$

since $J_{1/3}$ and $J_{-1/3}$ satisfy the same bounds for large z it suffices to estimate

$$\widetilde{A}(\theta) = 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{N+1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} J_{1/3}(\frac{2}{3} \nu\phi^{\frac{3}{2}}) dr.$$

It follows from (3) that,

$$J_{1/3}(\frac{2}{3}\nu\phi^{\frac{3}{2}}) = \sigma_1(\nu\phi^{\frac{3}{2}})e^{i\frac{2}{3}\nu\phi^{\frac{3}{2}}} + \sigma_2(\nu\phi^{\frac{3}{2}})e^{-i\frac{2}{3}\nu\phi^{\frac{3}{2}}},$$

where σ_i is a symbol of order $-\frac{1}{2}$. We therefore write $\tilde{A} = A_1 + A_2$, where

$$A_1(\theta) = 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) e^{i2^{j\beta}[\varphi(r,\theta)+2^{-j\beta}\frac{2}{3}\nu\phi^{\frac{3}{2}}]} (x\nu)^{\frac{N+1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_1(\nu\phi^{\frac{3}{2}}) dr,$$

$$A_2(\theta) = 2^{j(\alpha-N\beta)} \int_0^\infty \eta_3(x) \vartheta(r) e^{i2^{j\beta}[\varphi(r,\theta)-2^{-j\beta}\frac{2}{3}\nu\phi^{\frac{3}{2}}]} (x\nu)^{\frac{N+1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_2(\nu\phi^{\frac{3}{2}}) dr.$$

Let us first consider the integral A_1 , and now let $\tilde{\Phi}_\theta(r) = \varphi(r, \theta) + 2^{-j\beta}\frac{2}{3}\nu\phi^{\frac{3}{2}}$. We note that $\partial_r \tilde{\Phi}_\theta(r) = \partial_r \Phi_\theta(r)$. It therefore follows that $\tilde{\Phi}_\theta$ behaves exactly as Φ_θ did in §5.2 and so for C_β chosen small enough we again wish to integrate by parts. In this case our amplitude

$$\tilde{\Psi}_\theta(r) = \eta_3(x) \vartheta(r) (x\nu)^{\frac{N+1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_1(\nu\phi^{\frac{3}{2}}),$$

courtesy of the symbol estimates $|\partial_r^\ell \sigma_1(\nu\phi^{\frac{3}{2}})| \leq Cx^{\frac{1}{4}+\ell}(\nu-x)^{-\frac{3}{4}-\ell}$, satisfies for $\ell = 0, 1, \dots$ the differential inequality

$$|\partial_r^\ell \tilde{\Psi}_\theta(r)| \leq C(\nu-x)^{-\ell-\frac{1}{4}} \nu^{N-n+\frac{3}{4}+\ell}.$$

Integrating by parts N' times we therefore get the estimate

$$\begin{aligned} |A_1(\theta)| &\leq C2^{j(\alpha-N\beta)} \left| \int_0^\infty \tilde{\Psi}_\theta(r) e^{i2^{j\beta}\tilde{\Phi}_\theta(r)} dr \right| \\ &\leq C2^{j(\alpha-(N+N')\beta)} \int_{\nu^{\frac{1}{3}} \leq \nu-x} (\nu-x)^{-N'-\frac{1}{4}} \nu^{N+N'-\frac{1}{4}-n} dx \\ &\leq C2^{j(\alpha-(N+N')\beta)} \nu^{N+\frac{2}{3}N'-n} \\ &\leq C2^{j(\alpha-n\beta-\frac{N'}{3}\beta)}, \end{aligned}$$

again provided $N > n$, but also that $N' \geq 1$. We of course obtain the same estimate for A_2 since its phase is trivially never stationary. In *Region A* we therefore obtain the estimate

$$|\mu_{j,3}(\kappa, \lambda 2^{2j})| \leq C2^{j(\alpha-(n+\frac{1}{2})\beta)}.$$

Region B: $\lambda \geq C_\beta 2^{j\beta}$. Recall that $\lambda r^2 \sin \theta \sim \nu$ and hence $\sin \theta \leq C\nu\lambda^{-1}$. Trivially we get the estimate $|\mu_j(\kappa, \lambda 2^{2j})| \leq C2^{j\alpha} \lambda^{-n} \nu^{\frac{1}{2}}$, we'll use the oscillation to do better. Using the Airy asymptotic form (7) we may write $\mathcal{J}_3 = c\mathcal{A} + \mathcal{E}_A$, where

$$\mathcal{A}(r) = 2^{j\alpha} \int_0^\pi \eta_3(x) (\sin \theta)^{n-1} e^{i\lambda r^2 \cos \theta} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{\phi'}\right)^{\frac{1}{2}} \text{Ai}(-\nu^{\frac{2}{3}}\phi) d\theta.$$

ERROR TERM:

$$\begin{aligned}
\int_0^\infty \vartheta(r)r^{-1-\alpha}|\mathcal{E}_A(r)|dr &\leq C2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \int_{\frac{1}{2}}^2 (x\nu)^{\frac{1-n}{2}} x^{-\frac{3}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{\phi'}\right)^{\frac{1}{2}} |\widetilde{\text{Ai}}(-\nu^{\frac{2}{3}}\phi)| dr d\theta \\
&\leq C2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \int_{\frac{1}{2}}^2 (x\nu)^{-\frac{1}{2}(n+\frac{1}{3})} (-\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} (-\nu\phi^{\frac{3}{2}})^{-\frac{1}{2}} dr d\theta \\
&\leq C2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \int_{\frac{1}{2}}^2 (x\nu)^{-\frac{1}{2}(n+\frac{1}{2})} \nu^{\frac{1}{4}} (\nu-x)^{-\frac{1}{4}} dr d\theta \\
&\leq C2^{j\alpha} (\lambda\nu)^{-\frac{1}{2}(n+\frac{1}{2})} \int_{\sin \theta \leq C\nu\lambda^{-1}} (\sin \theta)^{\frac{1}{2}(n-\frac{5}{2})} \nu^{-\frac{3}{4}} \int_{\nu-x \leq \nu} (\nu-x)^{-\frac{1}{4}} dx d\theta \\
&\leq C2^{j\alpha} \lambda^{-n} \nu^{-\frac{1}{2}}.
\end{aligned}$$

MAIN TERM: As in *Region A* it suffices to estimate

$$\widetilde{\mathcal{A}}(r) = 2^{j\alpha} \int_0^\pi \eta_3(x)(\sin \theta)^{n-1} e^{i\lambda r^2 \cos \theta} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} J_{1/3}(\frac{2}{3}\nu\phi^{\frac{3}{2}}) d\theta,$$

and since $\nu\phi^{\frac{3}{2}} \geq 1$ we shall, as before, write $\widetilde{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1(r) = 2^{j\alpha} \int_0^\pi \eta_3(x)(\sin \theta)^{n-1} e^{i[\lambda r^2 \cos \theta + \frac{2}{3}\nu\phi^{\frac{3}{2}}]} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_1(\nu\phi^{\frac{3}{2}}) d\theta,$$

$$\mathcal{A}_2(r) = 2^{j\alpha} \int_0^\pi \eta_3(x)(\sin \theta)^{n-1} e^{i[\lambda r^2 \cos \theta - \frac{2}{3}\nu\phi^{\frac{3}{2}}]} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_2(\nu\phi^{\frac{3}{2}}) d\theta.$$

Let us first consider the integral \mathcal{A}_1 , and now let $\widetilde{\Phi}_r(\theta) = \lambda r^2 \cos \theta + \frac{2}{3}\nu\phi^{\frac{3}{2}}$. As above we note that $\partial_\theta \widetilde{\Phi}_r(\theta) = \partial_\theta \Phi_r(\theta)$ and thus

$$\partial_\theta \widetilde{\Phi}_r(\theta) = -\frac{1}{2}\lambda r^2 \left(2 \sin \theta - \left(\frac{\nu-x}{x}\right)^{\frac{1}{2}} \cos \theta\right).$$

Therefore, since $\nu-x \leq x$ in this interval, it easy to see that

$$|\partial_\theta \widetilde{\Phi}_r(\theta)| \geq C\lambda,$$

provided $\cos \theta \leq \sin \theta$. With this in mind we shall this time write $\mathcal{A}_1 = \mathcal{A}_1^1 + \mathcal{A}_1^2$, where

$$\mathcal{A}_1^1(r) = 2^{j\alpha} \int_0^\pi (1 - \chi(\frac{1}{10} \tan \theta)) \widetilde{\Psi}_r(\theta) e^{i\widetilde{\Phi}_r(\theta)} d\theta,$$

$$\mathcal{A}_1^2(r) = 2^{j\alpha} \int_0^\pi \chi(\frac{1}{10} \tan \theta) \widetilde{\Psi}_r(\theta) e^{i\widetilde{\Phi}_r(\theta)} d\theta,$$

and

$$\widetilde{\Psi}_r(\theta) = (\sin \theta)^{n-1} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\frac{1}{\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_1(\nu\phi^{\frac{3}{2}}).$$

Case 1: If $\tan \theta \geq 1$, then letting $\widetilde{\Psi}_r^1(\theta) = (1 - \chi(\frac{1}{10} \tan \theta)) \widetilde{\Psi}_r(\theta)$ we see that

$$|\widetilde{\Psi}_r^1(\theta)| \leq C\nu^{\frac{3}{4}-n} (\nu-x)^{-\frac{1}{4}},$$

$$|\partial_\theta \widetilde{\Psi}_r^1(\theta)| \leq C\nu^{\frac{7}{4}-n} (\nu-x)^{-\frac{5}{4}}.$$

Letting $y = x(\lambda, r, \frac{\pi}{2})$ we see that an application of van der Corput's lemma gives,

$$\begin{aligned} \int_0^\infty \vartheta(r)r^{-1-\alpha}|\mathcal{A}_1^1(r)|dr &\leq C2^{j\alpha}\lambda^{-1}\left(\int_{\frac{1}{2}}^2|\tilde{\Psi}_r^1(\frac{\pi}{2})|dr + \int_0^\pi\int_{\frac{1}{2}}^2|\partial_\theta\tilde{\Psi}_r^1(\theta)|drd\theta\right) \\ &\leq C2^{j\alpha}\lambda^{-1}\nu^{-\frac{1}{4}-n}\left(\int_{\nu-y\leq\nu}(\nu-y)^{-\frac{1}{4}}dy + \nu\int_0^\pi\int_{\nu^{\frac{1}{3}}\leq\nu-x\leq\nu}(\nu-x)^{-\frac{5}{4}}dx d\theta\right) \\ &\leq C2^{j\alpha}\lambda^{-1}\nu^{-\frac{1}{4}-n}\left(\nu^{\frac{3}{4}} + \nu^{\frac{11}{12}}\right) \\ &\leq C2^{j\alpha}\lambda^{-(n+\frac{1}{3})}. \end{aligned}$$

Here we have use the fact that $\nu \geq C\lambda$, this is a consequence of $\sin \theta \geq \frac{1}{2}$.

Case 2: If $\tan \theta \leq 1$, then letting $\tilde{\Psi}_r^2(\theta) = \chi(\frac{1}{10}\tan \theta)\tilde{\Psi}_r(\theta)$ we see that

$$|\tilde{\Psi}_r^2(\theta)| \leq C(\sin \theta)^{n-1}\nu^{\frac{3}{4}-n}(\nu-x)^{-\frac{1}{4}} \leq C\lambda^{-(n-1)}\nu^{-\frac{1}{4}}(\nu-x)^{-\frac{1}{4}},$$

$$\begin{aligned} |\partial_\theta\tilde{\Psi}_r^2(\theta)| &\leq C(\sin \theta)^{n-2}\nu^{\frac{3}{4}-n}(\nu-x)^{-\frac{1}{4}} + C(\sin \theta)^{n-1}\nu^{\frac{2}{3}-n}|\partial_\theta[(\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}}\sigma_1(\nu\phi^{\frac{3}{2}})]| \\ &\leq C\lambda^{-(n-2)}\nu^{-\frac{5}{4}}(\nu-x)^{-\frac{1}{4}} + C\lambda^{-(n-1)}\nu^{-\frac{1}{3}}\partial_\theta(\nu\phi^{\frac{3}{2}})^{-\frac{1}{6}}. \end{aligned}$$

Here of course our phase may be stationary, however it is easy to see that

$$|\partial_\theta^2\tilde{\Phi}_r(\theta)| \geq C\lambda^2(\nu-x)^{-\frac{1}{2}}x^{-\frac{1}{2}} \geq \lambda^2\nu^{-1}.$$

Arguing as above we see that van der Corput's lemma gives,

$$\begin{aligned} \int_0^\infty \vartheta(r)r^{-1-\alpha}|\mathcal{A}_1^2(r)|dr &\leq C2^{j\alpha}\lambda^{-1}\nu^{\frac{1}{2}}\int_0^\pi\int_{\frac{1}{2}}^2|\partial_\theta\tilde{\Psi}_r^2(\theta)|drd\theta \\ &\leq C2^{j\alpha}\lambda^{-n}\left(\lambda\nu^{-1}\int_{\theta\leq\frac{\nu}{\lambda}}d\theta + \nu^{\frac{1}{6}}\int_{\frac{1}{2}}^2\int_0^\pi\partial_\theta(\nu\phi^{\frac{3}{2}})^{-\frac{1}{6}}d\theta dr\right) \\ &\leq C2^{j\alpha}\lambda^{-n}\left(\lambda\nu^{-1}\int_{\theta\leq\frac{\nu}{\lambda}}d\theta + \nu^{\frac{1}{4}}\int_{\frac{1}{2}}^2(\nu-y)^{-\frac{1}{4}}dr\right) \\ &\leq C2^{j\alpha}\lambda^{-n}. \end{aligned}$$

The integral \mathcal{A}_2 can be treated in exactly the same way, thus in *Region B* we obtain

$$|\mu_{j,3}(\kappa, \lambda 2^{2j})| \leq C2^{j\alpha}\lambda^{-n}.$$

5.4. Neighborhood of the turning point: $|\nu-x| \leq \nu^{\frac{1}{3}}$. Here we just use a size estimate and the fact that $|\Lambda_\kappa^\delta(x)| \leq C\nu^{-\frac{1}{3}}$. This is the best we can do since $\nu\phi^{\frac{3}{2}} \leq \nu(\frac{\nu-x}{\nu})^{\frac{3}{2}} \leq 1$. Notice again that $(\lambda\nu)^{\frac{1}{2}} \leq 2\lambda$.

Region A: $\lambda \leq C_\beta 2^{j\beta}$. We should integrate by parts and since $x \leq C(x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$\begin{aligned} |I_4(\theta)| &= 2^{j(\alpha-N\beta)} \left| \int_0^\infty \eta_4(x) \vartheta(r) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{N+1-n}{2}} \Lambda_{\kappa-N}^{n-1+N}(x) dr \right| \\ &\leq C 2^{j(\alpha-N\beta)} \nu^{N-n} \int_{|\nu-x| \leq \nu^{\frac{1}{3}}} \nu^{-\frac{1}{3}} dx \\ &\leq C 2^{j(\alpha-n\beta)}, \end{aligned}$$

provided $N > n$. So in *Region A* we obtain the estimate

$$|\mu_{j,4}(\kappa, \lambda 2^{2j})| \leq C 2^{j(\alpha-n\beta)}.$$

Region B: $\lambda \geq C_\beta 2^{j\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\begin{aligned} |\mu_{j,4}(\kappa, \lambda 2^{2j})| &= 2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \left| \int_{\frac{1}{2}}^2 \eta_4(x) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr \right| d\theta \\ &\leq C 2^{j\alpha} \int_{\sin \theta \leq \frac{\nu}{\lambda}} (\sin \theta)^{n-1} \nu^{-n} \int_{|\nu-x| \leq \nu^{\frac{1}{3}}} \nu^{-\frac{1}{3}} dx d\theta \\ &\leq C 2^\alpha \lambda^{-n}. \end{aligned}$$

5.5. Monotonic region I: $\nu + \nu^{\frac{1}{3}} \leq x \leq \frac{3}{2}\nu$. Recall that here $|\Lambda_\kappa^\delta(x)| \leq C \nu^{-\frac{1}{4}} (x-\nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}} (x-\nu)^{\frac{3}{2}}}$. Notice that in this interval $(\lambda\nu)^{\frac{1}{2}} \leq \lambda$.

Region A: $\lambda \leq C_\beta 2^{j\beta}$. We should integrate by parts and since $x \leq C(x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$\begin{aligned} |I_5(\theta)| &= 2^{j(\alpha-N\beta)} \left| \int_0^\infty \eta_5(x) \vartheta(r) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{N+1-n}{2}} \Lambda_{\kappa-N}^{n-1+N}(x) dr \right| \\ &\leq C 2^{j(\alpha-N\beta)} \nu^{N-n} \int_{x-\nu \geq \nu^{\frac{1}{3}}} \nu^{-\frac{1}{4}} (x-\nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}} (x-\nu)^{\frac{3}{2}}} dx \\ &\leq C 2^{j(\alpha-n\beta)} \int_{u \geq 1} u^{-\frac{1}{4}} e^{-\gamma_1 u^{\frac{3}{2}}} du \\ &\leq C 2^{j(\alpha-n\beta)}, \end{aligned}$$

again provided N is large enough. So we again in *Region A* obtain the estimate

$$|\mu_{j,5}(\kappa, \lambda 2^{2j})| \leq C 2^{j(\alpha-n\beta)}.$$

Region B: $\lambda \geq C_\beta 2^{j\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\begin{aligned} |\mu_{j,5}(\kappa, \lambda 2^{2j})| &= 2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \left| \int_{\frac{1}{2}}^2 \eta_5(x) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr \right| d\theta \\ &\leq C 2^{j\alpha} \int_{\sin \theta \leq \frac{\nu}{\lambda}} (\sin \theta)^{n-1} \nu^{-n} \int_{x-\nu \geq \nu^{\frac{1}{3}}} \nu^{-\frac{1}{4}} (x-\nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}} dx d\theta \\ &\leq C 2^{j\alpha} \int_{\sin \theta \leq \frac{\nu}{\lambda}} (\sin \theta)^{n-1} \nu^{-n} \int_{u \geq 1} u^{-\frac{1}{4}} e^{-\gamma_1 u^{\frac{3}{2}}} du d\theta \\ &\leq C 2^{j\alpha} \lambda^{-n}. \end{aligned}$$

5.6. Monotonic region II: $x \geq \frac{3\nu}{2}$. Here we have the trivial estimate $|\Lambda_\kappa^\delta(x)| \leq C e^{-\gamma_2 x}$. Again in this interval $(\lambda\nu)^{\frac{1}{2}} \leq \lambda$.

Region A: $\lambda \leq C_\beta 2^{j\beta}$. We should integrate by parts and since $x \geq (x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$\begin{aligned} |I_6(\theta)| &= 2^{j(\alpha-N\beta)} \left| \int_0^\infty \eta_6(x) \vartheta(r) e^{i2^{j\beta}\varphi(r,\theta)} x^N (x\nu)^{\frac{1-n}{2}} \Lambda_{\kappa-N}^{n-1+N}(x) dr \right| \\ &\leq C 2^{j(\alpha-N\beta)} \nu^{\frac{1-n}{2}} \int_{x \geq \nu} x^{N-1-\frac{n-1}{2}} e^{-\gamma_2 x} dx \\ &\leq C 2^{j(\alpha-N\beta)}, \end{aligned}$$

for all $N \geq 0$. So in *Region A* we obtain the estimate

$$|\mu_{j,6}(\kappa, \lambda 2^{2j})| \leq C 2^{j(\alpha-N\beta)}.$$

Region B: $\lambda \geq C_\beta 2^{j\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\begin{aligned} |\mu_{j,6}(\kappa, \lambda 2^{2j})| &= 2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \left| \int_{\frac{1}{2}}^2 \eta_6(x) e^{i2^{j\beta}\varphi(r,\theta)} (x\nu)^{\frac{1-n}{2}} \Lambda_\kappa^{n-1}(x) dr \right| d\theta \\ &\leq C 2^{j\alpha} \int_0^\pi (\sin \theta)^{n-1} \nu^{-n} \int_{x \geq \nu} e^{-\gamma_2 x} dx d\theta \\ &\leq C 2^{j\alpha} \lambda^{-N} \nu^{-n} \int_{\sin \theta \geq \frac{\nu}{\lambda}} (\sin \theta)^{n-1-N} d\theta \\ &\leq C 2^{j\alpha} \lambda^{-n} \nu^{-N}. \end{aligned}$$

We have therefore established the Theorem 5.1.

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