A CLASS OF STRONGLY SINGULAR RADON TRANSFORMS ON THE HEISENBERG GROUP

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Abstract. We primarily consider here the L^2 mapping properties of a class of strongly singular Radon transforms on the Heisenberg group $\mathbf{H}^n$; these are convolution operators on $\mathbf{H}^n$ with kernels of the form $M(z,t) = K(z)\delta_0(t)$ where K is a strongly singular kernel on $\mathbf{C}^n$. Our results are obtained by utilizing the group Fourier transform and uniform asymptotic forms for Laguerre functions due to Erdélyi.

We also discuss the behavior of related twisted strongly singular operators on $L^2(\mathbf{C}^n)$ and obtain results in this context independently of group Fourier transform methods. Key to this argument is a generalization of the results for classical strongly singular integrals on $L^2(\mathbf{R}^d)$.

1. Introduction

In this article our main consideration shall be the L^2 mapping properties of a class of strongly singular Radon transforms on the Heisenberg group $\mathbf{H}^n$. More precisely we consider convolution operators on $\mathbf{H}^n$ with kernels of the form $M(z,t) = K(z)\delta_0(t)$ where K is a kernel on $\mathbf{C}^n$ that is too singular at the origin to be of Calderón-Zygmund type and has this strong singularity compensated for by the introduction of a suitably large oscillation. Our main result is stated in §1.2 below and is obtained by utilizing group Fourier transform methods and uniform asymptotic forms for Laguerre functions due to Erdélyi [3].

We also discuss the behavior of related twisted strongly singular integral operators on $L^2(\mathbf{C}^n)$, these results are stated in §1.3 and are obtained independently of group Fourier transform techniques. Key to these arguments is a generalization of existing results for classical strongly singular integrals on $L^2(\mathbf{R}^d)$. We choose to state these results first.

1.1. Strongly singular integrals on $\mathbf{R}^d$.

These are operators T, initially defined as mappings from test functions in $\mathcal{S}(\mathbf{R}^d)$ to distributions in $\mathcal{S}'(\mathbf{R}^d)$, to which are associated kernels $K_{\alpha,\beta}(x,y)$, defined when $x \neq y$, that take the form

$$K_{\alpha,\beta}(x,y) = a(x,y)e^{i\varphi(x,y)}. \tag{1}$$

We assume that the amplitude and phase satisfy the differential inequalities

$$|\partial_x^\mu \partial_y^\nu a(x,y)| \leq C_{\mu,\nu}|x-y|^{-d-\alpha-|\mu|-|\nu|} \tag{2a}$$

$$|\partial_x^\mu \partial_y^\nu \varphi(x,y)| \leq C_{\mu,\nu}|x-y|^{-\beta-|\mu|-|\nu|}, \tag{2b}$$

that φ is real-valued and furthermore that

$$|\nabla_x\varphi(x,y)|,\ |\nabla_y\varphi(x,y)| \geq C|x-y|^{-\beta-1} \tag{2c}$$

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with $\alpha \geq 0$ and $\beta > 0$. In the case where $\alpha = 0$ we must make the further assumption that our amplitude a is compactly supported in a neighborhood of the diagonal $x = y$, this is of course also the only region of any interest when $\alpha > 0$.

It is clear that the estimates (2) also hold uniformly for the dilated functions

$$a_\lambda(x, y) = \lambda^{d+\alpha} a(\lambda x, \lambda y) \quad \text{and} \quad \varphi_\lambda(x, y) = \lambda^\beta \varphi(\lambda x, \lambda y),$$

and in additions to the differential inequalities (2) above we also make the following *non-degeneracy* assumption, namely that

$$\left| \det \left(\frac{\partial^2 \varphi_\lambda(x, y)}{\partial x_i \partial y_j} \right) \right| \geq C > 0 \tag{3}$$

uniformly in λ, such kernels we shall call *(non-degenerate) strongly singular integral kernels.*

Our strongly singular integral operators T are related to these kernels $K_{\alpha,\beta}$ as follows: for $f \in \mathcal{S}$ with compact support we identify the distribution Tf with the function

$$Tf(x) = \int K_{\alpha,\beta}(x, y) f(y)\, dy, \tag{4}$$

for x is outside the support of f. Our result for such operators is the following.

Theorem 1.1. *The operator T, initially given by (4), extends to a bounded operator on $L^2(\mathbf{R}^d)$ if and only if $\alpha \leq \frac{d\beta}{2}$.*

The proof of this result, amongst other things, is essentially contained in §5; see also [8].

The model case for operators of this type are those with kernels

$$K_{\alpha,\beta}(x, y) = \widetilde{K}_{\alpha,\beta}(x - y)$$

where $\widetilde{K}_{\alpha,\beta}$ is a distribution[1] on $\mathbf{R}^d$ that away from the origin agrees with the *radial* function

$$\widetilde{K}_{\alpha,\beta}(x) = |x|^{-d-\alpha} e^{i|x|^{-\beta}} \chi(|x|), \tag{5}$$

again $\beta > 0$ and here χ is smooth and compactly supported in a small neighborhood of the origin.

Operators of this type were first studied by Hirschman [7] in the case $d = 1$ and then in higher dimensions by Wainger [13], Fefferman [4], and Fefferman and Stein [5]. That these operators are indeed of the form considered in Theorem 1.1, and that their kernels are in particular non-degenerate, is an easy exercise; see §5.2.

To establish Theorem 1.1 in this model case it is efficient to use Fourier transform methods. Since $\widetilde{K}_{\alpha,\beta}$ is a radial compactly supported distribution it is well known that its Fourier transform is a smooth radial function given by

$$m(\xi) = (2\pi)^{\frac{d}{2}} \int_0^\infty \chi(r) r^{-1-\alpha} e^{ir^{-\beta}} J_{\frac{d-2}{2}}(r|\xi|)(r|\xi|)^{\frac{2-d}{2}}\, dr, \tag{6}$$

where $J_{\frac{d-2}{2}}$ is a Bessel function; see [11]. Using Plancherel's theorem and the asymptotics of Bessel functions it is then straightforward to establish Theorem 1.1 in this case.

[1] The distribution-valued function $\alpha \mapsto \widetilde{K}_{\alpha,\beta}$, initially defined for $\operatorname{Re}\alpha < 0$, continues analytically to all of $\mathbf{C}$.

1.2. **The Heisenberg group.** The Heisenberg group $\mathbf{H}^n$ is $\mathbf{C}^n \times \mathbf{R}$ endowed with the group law

$$[z,t] \cdot [w,s] = [z+w, t+s+\tfrac{1}{2}\operatorname{Im} z \cdot \bar{w}],$$

with identity the origin and inverses given by $[z,t]^{-1} = [-z,-t]$.

The following transformations are automorphisms of the group $\mathbf{H}^n$:

- *nonisotropic dilations* $[z,t] \mapsto \delta \circ [z,t] = [\delta z, \delta^2 t]$, for all $\delta > 0$;
- *rotations* $[z,t] \mapsto [Uz,t]$, with U a unitary transformation of $\mathbf{C}^n$.

The usual Lebesgue measure $dz\,dt$ on $\mathbf{C}^n \times \mathbf{R}$ is the Haar measure for $\mathbf{H}^n$.

A natural analogue, in this Heisenberg group setting, to the model operators discussed above has been studied by the author; see [9].

In this article we shall consider the class of strongly singular Radon transforms on the Heisenberg group $\mathbf{H}^n$ formally given by

$$Rf(z,t) = \int_{\mathbf{H}^n} M([w,s]^{-1} \cdot [z,t]) f(w,s)\,dw\,ds \tag{7}$$

and M are distribution kernels of the form

$$M = \widetilde{K}_{\alpha,\beta} \otimes \delta_0 \tag{8}$$

with $\widetilde{K}_{\alpha,\beta}$ the *radial* strongly singular kernels on $\mathbf{C}^n$ ($\mathbf{R}^d$ with $d = 2n$) given by (5).

Our main result is then the following.

Theorem 1.2. *R extends to a bounded operator on $L^2(\mathbf{H}^n)$ if and only if $\alpha \le (n - \frac{1}{6})\beta$.*

We obtain this result via group Fourier transform methods. Similar methods were employed by the author in [9] however the arguments in this setting are simpler.

1.3. **Twisted strongly singular integrals on $\mathbf{C}^n$.** The operators above are of course intimately connected with the *twisted convolution* operators

$$R^\lambda f(z) = \int_{\mathbf{C}^n} \widetilde{K}_{\alpha,\beta}(z-w) e^{i\frac{1}{2}\lambda \operatorname{Im} z \cdot \bar{w}} f(w) dw. \tag{9}$$

In fact it follows from taking the partial Fourier transform in the t variable and applying Plancherel's theorem that the boundedness of R on $L^2(\mathbf{H}^n)$ is formally equivalent to the uniform boundedness of R^λ on $L^2(\mathbf{C}^n)$ for $\lambda \neq 0$. It is therefore interesting to note the following corollary of the proof of Theorem 1.2; *for any fixed real λ, the operator R^λ extends to a bounded operator on $L^2(\mathbf{C}^n)$ whenever $\alpha \le n\beta$.*

However we shall prove a more general version of this result and do so independently of group Fourier transform methods.

We shall consider operators T^λ, initially defined as mappings from test functions in $\mathcal{S}(\mathbf{C}^n)$ to distributions in $\mathcal{S}'(\mathbf{C}^n)$, to which we associate *strongly singular integral kernels* $K_{\alpha,\beta}(z,w)$ on $\mathbf{C}^n$.

For $f \in \mathcal{S}$ with compact support we identify the distribution $T^\lambda f$ with the function

$$T^\lambda f(z) = \int_{\mathbf{C}^n} K_{\alpha,\beta}(z,w) e^{i\frac{1}{2}\lambda \operatorname{Im} z\cdot\bar{w}} f(w)dw, \tag{10}$$

for z is outside the support of f. Our result for such operators is the following.

Theorem 1.3. *For any λ real*

$$\|T^\lambda f\|_{L^2(\mathbf{C}^n)} \leq A_\lambda \|f\|_{L^2(\mathbf{C}^n)} \quad \textit{if and only if} \quad \alpha \leq n\beta.$$

Remark 1.4. (i) Theorem 1.1 is of course essentially a special case of Theorem 1.3 with $\lambda = 0$.
(ii) It would be of interest to know the precise behavior of the constant A_λ in Theorem 1.3 above as $|\lambda| \to \infty$.

In the next three sections we shall concern ourselves with the proof of Theorem 1.2: in §2 we introduce the group Fourier transform and reduce matters to basic Laguerre transform estimates; as with the model Euclidean case the asymptotics of special functions, in this case Laguerre polynomials, will be crucial and we include a discussion of these expansions in §3; while finally in §4 we present the proof of these key Laguerre transform estimates.

The proof Theorem 1.3 is presented in §5.

2. Reduction of Theorem 1.2 to Laguerre transform estimates

Let $\epsilon > 0$ and set $M^\epsilon(z,t) = \widetilde{K}^\epsilon_{\alpha,\beta}(z)\delta_0(t)$, where

$$\widetilde{K}^\epsilon_{\alpha,\beta}(z) = e^{-\epsilon|z|^{-\beta}} \widetilde{K}_{\alpha,\beta}(z)$$

and for $f \in L^2(\mathbf{H}^n)$ we define

$$R^\epsilon f(z,t) = f * M^\epsilon(z,t),$$

where convolution is taken with respect to the group structure on $\mathbf{H}^n$. It is then easy to see that if, for fixed $\epsilon > 0$, we integrate the function $\widetilde{K}^\epsilon_{\alpha,\beta}$ by parts N times and take the limit as $\epsilon \to 0$ then this must agree with the unique analytic continuation of $\widetilde{K}_{\alpha,\beta}$ to the half plane $\operatorname{Re}(\alpha) < N\beta$. It then follows that for $f \in \mathcal{S}(\mathbf{H}^n)$,

$$Rf(z,t) = \lim_{\epsilon\to 0} R^\epsilon f(z,t).$$

We shall therefore, in the following, content ourselves with studying the operator R^ϵ.

2.1. Group Fourier transform.

It follows from Plancherel's theorem for the group Fourier transform that

$$\|R^\epsilon f\|_{L^2(\mathbf{H}^n)} \leq A\|f\|_{L^2(\mathbf{H}^n)} \Leftrightarrow \|\widehat{M^\epsilon}(\lambda)\|_{Op} \leq A \text{ uniformly over } \lambda \neq 0,$$

where $\widehat{M^\epsilon}(\lambda)$ denotes the group Fourier transform of M^ϵ, which for each $\lambda \neq 0$ is an operator on the Hilbert space $L^2(\mathbf{R}^n)$. Now since M^ϵ were chosen *radial* on $\mathbf{H}^n$, i.e. $M^\epsilon(z,t) = M^\epsilon_0(|z|,t)$ for some function M^ϵ_0, it is a well known result of Geller [6] that the operators $\widehat{M^\epsilon}(\lambda)$ are in fact, for each $\lambda \neq 0$, diagonal with respect to a (rescaled) Hermite basis for $L^2(\mathbf{R}^n)$. More precisely

$$\widehat{M^\epsilon}(\lambda) = C_n\big(\delta_{\mathbf{j},\mathbf{k}}\ \mu(|\mathbf{k}|,\lambda)\big)_{\mathbf{j},\mathbf{k}\in\mathbf{N}^n},$$

where C_n is a constant which depends only on the dimension and the diagonal entries $\mu(|\mathbf{k}|, \lambda)$ can be expressed explicitly in terms a Laguerre transform. Denoting $k = |\mathbf{k}|$ we in fact have

$$\mu(k,\lambda) = c_k^{n-1} \int_0^\infty e^{-\epsilon r^{-\beta}} \chi(r) r^{-1-\alpha} e^{ir^{-\beta}} \Lambda_k^{n-1}(\tfrac{1}{2}|\lambda| r^2)(\tfrac{1}{2}|\lambda| r^2)^{\frac{1-n}{2}}\, dr, \tag{11}$$

where $c_k^\delta = \left(\frac{k!}{(k+\delta)!}\right)^{1/2}$ and $\Lambda_k^\delta(x)$ is a Laguerre function of type δ. Recall that Laguerre functions of type δ, $\delta > -1$, form an orthonormal basis for $L^2(\mathbf{R}^+)$ and are given by $\Lambda_k^\delta(x) = c_k^\delta L_k^\delta(x) e^{-\frac{1}{2}x} x^{\frac{\delta}{2}}$, where $L_k^\delta(x) = \sum_{j=0}^k \binom{k+\delta}{k-j} \frac{(-x)^j}{j!}$ are the Laguerre polynomials of type δ.

It therefore follows that the operators $\widehat{M^\epsilon}(\lambda)$ are bounded on $L^2(\mathbf{R}^n)$ if and only if the diagonal scalars $\mu(k,\lambda)$ are bounded uniformly in k, and hence

$$\|R^\epsilon f\|_{L^2(\mathbf{H}^n)} \le A\|f\|_{L^2(\mathbf{H}^n)} \Leftrightarrow |\mu(k,\lambda)| \le A', \text{ uniformly in } k \text{ and } \lambda \ne 0.$$

For more on the group Fourier transform and Laguerre functions see [10] and [12].

2.2. **Main estimates.** We have seen that matters reduce to the study of the 'Fourier transforms' $\mu(k,\lambda)$. Our main estimate is then the following.

Theorem 2.1. (1) *If $|\lambda| \le k$, then $|\mu(k,\lambda)| \le c_0(1+|\lambda|k)^{\frac{\alpha-n\beta}{2(\beta+1)}}$,*
(2) *If $|\lambda| \ge k$, then as $k \to \infty$*

$$\mu(k,\lambda) = c_1(|\lambda|k)^{\frac{\alpha-(n-\frac{1}{6})\beta}{2(\beta+1)}} e^{ic_2(|\lambda|k)^{\frac{\beta}{2(\beta+1)}}} + O\Big((|\lambda|k)^{\frac{\alpha-n\beta}{2(\beta+1)}}\Big)$$

where the constants c_0, c_1, and c_2 above are independent of k and λ.

It is clear from the remarks above that Theorem 1.2 will be an immediate consequence of Theorem 2.1, we present the proof of Theorem 2.1 in §4.

3. Asymptotic properties of Laguerre functions

Recall that Laguerre functions of type δ, $\delta > -1$, form an orthonormal basis for $L^2(\mathbf{R}^+)$ and are given by

$$\Lambda_k^\delta(x) = \left(\tfrac{k!}{(k+\delta)!}\right)^{1/2} L_k^\delta(x) e^{-\frac{1}{2}x} x^{\frac{\delta}{2}},$$

where $L_k^\delta(x) = \sum_{j=0}^k \binom{k+\delta}{k-j} \frac{(-x)^j}{j!}$ are the Laguerre polynomials of type δ.

The two asymptotic formulae below which hold uniformly in their respective ranges of validity (which overlap) are due to Erdélyi [3]. In what follows $\nu = 4k + 2\delta + 2$ and $N = \nu/4$.

3.1. **The Bessel asymptotic forms.** Let $0 \le x \le b\nu$, $b < 1$. Then for $k \ge k_0$,

$$\Lambda_k^\delta(x) = \Big(\frac{(\delta+k)!}{k!}\Big)^{\frac{1}{2}} 2^{\delta-\frac{1}{2}} \nu^{-\frac{\delta}{2}} \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} \{J_\delta(\nu\psi) + O[\nu^{-1}(\tfrac{x}{\nu-x})^{\frac{1}{2}} \widetilde{J}_\delta(\nu\psi)]\},$$

and so

$$\Lambda_k^\delta(x) = C_1(\delta) \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} \{J_\delta(\nu\psi) + O[\nu^{-1}(\tfrac{x}{\nu-x})^{\frac{1}{2}} \widetilde{J}_\delta(\nu\psi)]\} \tag{12}$$

where $C_1(\delta)$ is a constant independent of k, $\psi=\psi(t)$ satisfies

$$\psi'(t)=\frac{1}{2}\Big(\frac{1}{t}-1\Big)^{\frac{1}{2}} \tag{13}$$

and $t=\frac{x}{\nu}$. For $0\leq t<1$,

$$\psi(t)=\tfrac{1}{2}[(t-t^2)^{\frac{1}{2}}+\sin^{-1}t^{\frac{1}{2}}],$$

and

$$\widetilde{J_\delta}(u)=\begin{cases}J_\delta(u) & \text{if } u \text{ sufficiently small,}\\ \big(|J_\delta(u)|^2+|Y_\delta(u)|^2\big)^{\frac{1}{2}} & \text{otherwise,}\end{cases}$$

here Y_δ and J_δ are Bessel functions of order δ.

Lemma 3.1. *If* $0\leq t\leq\frac{1}{2}$, *then* $\frac{1}{2}t^{\frac{1}{2}}\leq\psi(t)\leq t^{\frac{1}{2}}$.

Proof. Let $f(t)=(t-t^2)^{\frac{1}{2}}+\sin^{-1}t^{\frac{1}{2}}$, notice then that $f'(t)=\big(\frac{1-t}{t}\big)^{\frac{1}{2}}$. Now if $0\leq s\leq\frac{1}{2}$, we have $\frac{1}{2}s^{-\frac{1}{2}}\leq f'(s)\leq s^{-\frac{1}{2}}$, and so

$$\frac{1}{2}\int_0^t s^{-\frac{1}{2}}ds\leq\int_0^t f'(s)ds\leq\int_0^t s^{-\frac{1}{2}}ds$$

which implies $t^{\frac{1}{2}}\leq f(t)\leq 2t^{\frac{1}{2}}$, since $f(0)=0$. □

3.2. **The Airy asymptotic forms.** Let $0<a\nu\leq x$, $a>0$. Then for $k\geq k_0$,

$$\Lambda_k^\delta(x)=\frac{(-1)^k}{\big(k!(\delta+k)!\big)^{\frac{1}{2}}}2^{\frac{5}{6}}N^{N+\frac{1}{6}}e^{-N}x^{-\frac{1}{2}}\Big(\frac{\pi}{-\phi'}\Big)^{\frac{1}{2}}\{\mathrm{Ai}(-\nu^{\frac{2}{3}}\phi)+O[x^{-1}\widetilde{\mathrm{Ai}}(-\nu^{\frac{2}{3}}\phi)]\},$$

and so, using Stirling's formula

$$\Lambda_k^\delta(x)=C_2(\delta)(-1)^k\nu^{\frac{1}{6}}x^{-\frac{1}{2}}\Big(\frac{1}{-\phi'}\Big)^{\frac{1}{2}}\{\mathrm{Ai}(-\nu^{\frac{2}{3}}\phi)+O[x^{-1}\widetilde{\mathrm{Ai}}(-\nu^{\frac{2}{3}}\phi)]\} \tag{14}$$

where $C_2(\delta)$ is a constant independent of k, $\phi=\phi(t)$ satisfies

$$[\phi(t)]^{\frac{1}{2}}\phi'(t)=\frac{1}{2}\Big(\frac{1}{t}-1\Big)^{\frac{1}{2}}, \tag{15}$$

and again $t=\frac{x}{\nu}$. Now one can show

$$\phi(t)=\Big(\frac{3}{4}\Big)^{\frac{2}{3}}\begin{cases}[\cos^{-1}t^{\frac{1}{2}}-(t-t^2)^{\frac{1}{2}}]^{\frac{2}{3}} & \text{if } 0<t\leq 1,\\ -[(t-t^2)^{\frac{1}{2}}-\cosh^{-1}t^{\frac{1}{2}}]^{\frac{2}{3}} & \text{if } t>1,\end{cases}$$

and

$$\widetilde{\mathrm{Ai}}(z)=\begin{cases}\mathrm{Ai}(z) & \text{if } z\geq 0,\\ \big(|\,\mathrm{Ai}(z)|^2+|\,\mathrm{Bi}(z)|^2\big)^{\frac{1}{2}} & \text{if } z\leq 0,\end{cases}$$

here Ai and Bi are Airy integrals[2].

Lemma 3.2. *If* $\frac{1}{2}\leq t\leq 1$, *then* $\frac{1}{2}(1-t)\leq\phi(t)\leq 1-t$.

[2] $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are independent solutions of the differential equation $\frac{d^2y}{dz^2}=zy$ and have the integral representations $\mathrm{Ai}(z)=\frac{1}{\pi}\int_0^\infty\cos(\frac{1}{3}t^3+zt)dt$ and $\mathrm{Bi}(z)=\frac{1}{\pi}\int_0^\infty\{e^{\frac{1}{3}t^3+zt}+\sin(\frac{1}{3}t^3+zt)\}dt$.

Proof. Let $g(t) = \cos^{-1} t^{\frac{1}{2}} - (t-t^2)^{\frac{1}{2}}$, notice then that $g'(t) = -\left(\frac{1-t}{t}\right)^{\frac{1}{2}}$. Now if $\frac{1}{2} \le s \le 1$, we have $(1-s)^{\frac{1}{2}} \le -g'(s) \le 2(1-s)^{\frac{1}{2}}$, and so

$$\int_t^1 (1-s)^{\frac{1}{2}} ds \le -\int_t^1 g'(s) ds \le 2\int_t^1 (1-s)^{\frac{1}{2}} ds$$

which implies $\frac{2}{3}(1-t)^{\frac{3}{2}} \le g(t) \le \frac{4}{3}(1-t)^{\frac{3}{2}}$, since $g(1)=0$. □

Note also that, for $z > 0$

$$\mathrm{Ai}(-z) = \tfrac{1}{3} z^{\frac{1}{2}} [J_{1/3}(\tfrac{2}{3} z^{\frac{3}{2}}) + J_{-1/3}(\tfrac{2}{3} z^{\frac{3}{2}})]$$
$$\mathrm{Bi}(-z) = \left(\tfrac{z}{3}\right)^{\frac{1}{2}} [J_{1/3}(\tfrac{2}{3} z^{\frac{3}{2}}) + J_{-1/3}(\tfrac{2}{3} z^{\frac{3}{2}})].$$

3.3. **Bessel functions.** The *Bessel functions*, defined for real $k > -\frac{1}{2}$ by the formula

$$J_k(\lambda) = (\pi^{\frac{1}{2}} \Gamma(k+\tfrac{1}{2}))^{-1} \left(\frac{\lambda}{2}\right)^k \int_{-1}^1 e^{i\lambda t}(1-t^2)^{k-\frac{1}{2}} dt$$

are a model case for oscillatory integrals in one dimension and using this theory one can show that

$$J_k(\lambda) = \sigma_1(\lambda) e^{i\lambda} + \sigma_2(\lambda) e^{-i\lambda}, \tag{16}$$

where $|\sigma_i^{(\ell)}(\lambda)| \le c_\ell (1+\lambda)^{-\frac{1}{2}-\ell}$; see for example [14].

3.4. **Trivial Estimates.** It follows from the asymptotics above that for k large we have the following crude estimates for our Laguerre function; see Askey and Wainger [1].

$$|\Lambda_k^\delta(x)| \le C \begin{cases} (x\nu)^{\frac{\delta}{2}} & \text{if } 0 \le x \le \frac{1}{\nu}, \\ (x\nu)^{-\frac{1}{4}} & \text{if } \frac{1}{\nu} \le x \le \frac{\nu}{2}, \\ \nu^{-\frac{1}{4}}(\nu - x)^{-\frac{1}{4}} & \text{if } \frac{\nu}{2} \le x \le \nu - \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{3}} & \text{if } \nu - \nu^{\frac{1}{3}} \le x \le \nu + \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{4}}(x-\nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}} & \text{if } \nu + \nu^{\frac{1}{3}} \le x \le \frac{3\nu}{2}, \\ e^{-\gamma_2 x} & \text{if } x \ge \frac{3\nu}{2}, \end{cases}$$

where $\gamma_1, \gamma_2 > 0$ are fixed constants.

4. Proof of Theorem 2.1

It is natural to consider the cases for bounded and unbounded k separately. We now fix k_0 to be a large constant and note that for $k \le k_0$ one can easily verify, by integration by parts, that

$$|\mu(k,\lambda)| \le C(1+|\lambda|)^{-N},$$

for all $N \ge 0$.

Since we are now only interested in the case when k is large, our main object of study, namely $\mu(k,\lambda)$, is therefore essentially

$$\mathcal{I} = \int_0^\infty e^{-\epsilon r^{-\beta}} \chi(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} \Lambda_k^{n-1}(x) dr,$$

where $x = \frac{1}{2}|\lambda|r^2$. We can of course always integrate by parts in r, but we must take care of what happens when the derivative hits the amplitude of $\mathcal{I}$.

Recall that $\Lambda_k^\delta(x) = c_k^\delta L_k^\delta(x)e^{-\frac{1}{2}x}x^{\frac{\delta}{2}}$. Now since $\frac{d}{dx}L_k^\delta(x) = -L_{k-1}^{\delta+1}(x)$ and $c_{k-1}^{\delta+1} = k^{-\frac{1}{2}}c_k^\delta$ it follows that

$$\frac{d}{dx}\Lambda_k^\delta(x) = \tfrac{1}{2}(\tfrac{\delta}{x} - 1)\Lambda_k^\delta(x) - (\tfrac{k}{x})^{\frac{1}{2}}\Lambda_{k-1}^{\delta+1}(x).$$

Therefore, using the fact that $\partial_r x = \frac{2x}{r}$ we see that

$$\partial_r\Lambda_k^\delta(x) = -r^{-1}[(x-\delta)\Lambda_k^\delta(x) + 2(xk)^{\frac{1}{2}}\Lambda_{k-1}^{\delta+1}(x)].$$

If we instead take N derivatives it is easy to see that we get

$$\partial_r^N\Lambda_k^\delta(x) = r^{-N}[P_N(x)\Lambda_k^\delta(x) + (xk)^{\frac{1}{2}}P_{N-1}(x)\Lambda_{k-1}^{\delta+1}(x) + \cdots + (xk)^{\frac{N}{2}}P_0(x)\Lambda_{k-N}^{\delta+N}(x)],$$

that is

$$\partial_r^N\Lambda_k^\delta(x) = r^{-N}\sum_{\ell=0}^{N}(xk)^{\frac{\ell}{2}}P_{N-\ell}(x)\Lambda_{k-\ell}^{\delta+\ell}(x), \tag{17}$$

where $P_{N-\ell}(x)$ is some polynomial of degree $N-\ell$ in x.

We therefore see that integration by parts will, in general, only help us if

$$\max\{(x\nu)^{\frac{1}{2}}, x\} \leq C_1r^{-\beta}.$$

In order to estimate $\mathcal{I}$ we shall make use of the asymptotics for Laguerre functions presented in §3, it is then natural to consider six separate regions and write $\mathcal{I} = \mathcal{I}_1 + \cdots + \mathcal{I}_6$ where

$$\mathcal{I}_j = \int_0^\infty \chi_j^\epsilon(r)r^{-1-\alpha}e^{ir^{-\beta}}(x\nu)^{\frac{1-n}{2}}\Lambda_k^{n-1}(x)dr,$$

where $\chi_1^\epsilon(r) = e^{-\epsilon r^{-\beta}}\chi(r)\vartheta_j(x,\nu)$ and each $\vartheta_j(x,\nu)$ localizes smoothly to the jth interval indicated in §3.4.

4.1. **Neighborhood of 0: $0 \leq x \leq \frac{1}{\nu}$.** Notice that here $x\nu \leq 1$ and $|\Lambda_k^\delta(x)| \leq C(x\nu)^{\frac{\delta}{2}}$. Using our trivial estimate it is easy to see that

$$|\partial_r^N\Lambda_k^\delta(x)| \leq Cr^{-N}(x\nu)^{\frac{\delta}{2}+N},$$

therefore integrating by parts N times we obtain the estimate

$$\begin{aligned}|\mathcal{I}_1| &= \Big|\int_0^\infty \chi_1^\epsilon(r)r^{-1-\alpha}e^{ir^{-\beta}}(x\nu)^{\frac{1-n}{2}}\Lambda_k^{n-1}(x)dr\Big| \\ &\leq C\int_{r\leq\min\{1,(|\lambda|\nu)^{-\frac{1}{2}}\}} r^{-1-\alpha+N\beta}dr \\ &\leq C(|\lambda|\nu)^{\frac{\alpha-N\beta}{2}},\end{aligned}$$

for any $N \geq 0$.

4.2. **Oscillatory interval I: $\frac{1}{\nu} \le x \le \frac{\nu}{2}$.** Notice that here $\frac{\nu}{2} \le \nu - x < \nu$, and $|\Lambda_k^\delta(x)| \le C(x\nu)^{-\frac{1}{4}}$.

In this interval we shall make explicit use of the oscillation in the main term of our asymptotic expansion, which are in this case is given in terms of Bessel functions; see §3.1. We note here that from (13) and Lemma 3.1 it follows that

$$\psi' \sim \left(\frac{\nu}{x}\right)^{\frac{1}{2}} \quad \text{while} \quad \psi \sim \left(\frac{x}{\nu}\right)^{\frac{1}{2}}.$$

The following estimates are then immediate,

$$(\nu\psi)^{-\frac{1}{2}} \sim (x\nu)^{-\frac{1}{4}} \quad \text{and} \quad \partial_r[(\nu\psi)^{-\frac{1}{2}}] = -\frac{x}{r\nu^{\frac{3}{2}}}\psi^{-\frac{3}{2}}\psi' \sim -\frac{1}{r}(x\nu)^{-\frac{1}{4}},$$

$$\partial_r\nu\psi = \frac{1}{r}x^{\frac{1}{2}}(\nu - x)^{\frac{1}{2}} \sim \frac{1}{r}(x\nu)^{\frac{1}{2}} \quad \text{and} \quad \partial_r^2\nu\psi = -\frac{1}{r^2}\frac{x^{\frac{3}{2}}}{(\nu-x)^{\frac{1}{2}}} \sim \frac{1}{r^2}x^{\frac{3}{2}}\nu^{-\frac{1}{2}},$$

$$\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \le C \quad \text{and} \quad \left|\partial_r\left[\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}\right]\right| \le C\frac{1}{r}.$$

Case 1: $(x\nu)^{\frac{1}{2}} \le C_1 r^{-\beta}$. We should integrate by parts and since $x \le (x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$I_2 = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha+N\beta} e^{ir^{-\beta}} (x\nu)^{\frac{N+1-n}{2}} \Lambda_{k-N}^{n-1+N}(x)dr.$$

Using the Bessel asymptotic forms (12) we may write $I_2 = cB + E_B$, where

$$B = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha+N\beta} e^{ir^{-\beta}} (x\nu)^{\frac{N+1-n}{2}} \left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} J_{n-1+N}(\nu\psi)dr.$$

Error term:

$$\begin{aligned}
|E_B| &\le C\int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha+N\beta}(x\nu)^{\frac{N+1-n}{2}}\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}\nu^{-1}\left(\tfrac{x}{\nu-x}\right)^{\frac{1}{2}}|\widetilde{J}_{n-1+N}(\nu\psi)|dr \\
&\le C\int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha+N\beta}(x\nu)^{\frac{1}{2}(N-n-\frac{1}{2})}dr \\
&\le C(|\lambda|\nu)^{\frac{1}{2}(N-n-\frac{1}{2})}\int\limits_{r^{\beta+1}\le(|\lambda|\nu)^{-\frac{1}{2}}} r^{-1-\alpha+N(\beta+1)-(n+\frac{1}{2})}dr \\
&\le C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}},
\end{aligned}$$

provided N is chosen large enough.

Main term: Since $\nu\psi \ge (x\nu)^{\frac{1}{2}} \ge 1$ it follows from (16) that

$$J_{n-1+N}(\nu\psi) = \sigma_1(\nu\psi)e^{i\nu\psi} + \sigma_2(\nu\psi)e^{-i\nu\psi},$$

where each σ_i is a symbol of order $-\frac{1}{2}$. Since $\partial_r\psi \sim \frac{1}{r}\psi$ it follows that

$$|\partial_r^\ell\sigma_i(\nu\psi)| \le Cr^{-\ell}(\nu\psi)^{-\frac{1}{2}}.$$

Write $B = B_1 + B_2$, where

$$B_1 = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha+N\beta} e^{i[r^{-\beta}+\nu\psi]}(x\nu)^{\frac{N+1-n}{2}}\left(\frac{\nu}{x}\right)^{\frac{1}{2}}\left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}}\sigma_1(\nu\psi)dr,$$

$$B_2 = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha+N\beta} e^{i[r^{-\beta}+\nu\psi]} (x\nu)^{\frac{N+1-n}{2}} \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} \sigma_2(\nu\psi) dr.$$

Let us first consider the integral B_1, and let $\varphi(r) = r^{-\beta} + \nu\psi$. Now

$$\partial_r \varphi(r) = -\beta r^{-(\beta+1)} + \tfrac{1}{r} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}},$$

and so if C_1 is chosen small enough it follows that

$$|\partial_r \varphi(r)| \geq C r^{-(\beta+1)}.$$

In addition to this we also have

$$\partial_r^2 \varphi(r) = \beta(\beta+1) r^{-(\beta+2)} - \tfrac{1}{r^2} \tfrac{x^{\frac{3}{2}}}{(\nu-x)^{\frac{1}{2}}} \geq C r^{-(\beta+2)}.$$

If we let

$$a(r) = \chi_2^\epsilon(r) r^{-1-\alpha+N\beta} (x\nu)^{\frac{N+1-n}{2}} \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} \sigma_1(\nu\psi),$$

then for all $\ell = 0, 1, \dots$ we have

$$|\partial_r^\ell a(r)| \leq C r^{-1-\alpha+N\beta-\ell} (x\nu)^{\frac{N}{2}} (\nu\psi)^{-\frac{1}{2}}.$$

Applying van der Corput's lemma (integration by parts) gives

$$|B_1| \leq C \int\limits_{r^{\beta+1} \leq (|\lambda|\nu)^{-\frac{1}{2}}} r^{-1-\alpha+(N+1)\beta} (x\nu)^{\frac{2N+1-2n}{4}} dr \leq C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}},$$

provided N is chosen large enough as before.

Of course the phase in B_2 is never stationary, so we obtain the same estimate for B_2.

Case 2: $(x\nu)^{\frac{1}{2}} \geq C_1 r^{-\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\mathcal{I}_2 = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} \Lambda_k^{n-1}(x) dr.$$

Using the asymptotic forms (12) we may write $\mathcal{I}_2 = c\mathcal{B} + \mathcal{E}_B$, where

$$\mathcal{B} = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} J_{n-1}(\nu\psi) dr.$$

Error term:

$$\begin{aligned} |\mathcal{E}_B| &\leq C \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha} (x\nu)^{\frac{1-n}{2}} \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} \nu^{-1} \Big(\frac{x}{\nu - x}\Big)^{\frac{1}{2}} |\widetilde{J}_{n-1}(\nu\psi)| dr \\ &\leq C \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha} (x\nu)^{-\frac{1}{2}(n+\frac{1}{2})} dr \\ &\leq C(|\lambda|\nu)^{-\frac{1}{2}(n+\frac{1}{2})} \int\limits_{r^{\beta+1} \geq (|\lambda|\nu)^{-\frac{1}{2}}} r^{-1-\alpha-(n+\frac{1}{2})} dr \\ &\leq C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}}. \end{aligned}$$

MAIN TERM: As above we shall use (16) and write $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$, where

$$\mathcal{B}_1 = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha} e^{i[r^{-\beta}+\nu\psi]} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} (x\nu)^{\frac{1-n}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \sigma_1(\nu\psi) dr,$$

$$\mathcal{B}_2 = \int_0^\infty \chi_2^\epsilon(r) r^{-1-\alpha} e^{i[r^{-\beta}-\nu\psi]} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} (x\nu)^{\frac{1-n}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \sigma_2(\nu\psi) dr.$$

Let us first consider the integral $\mathcal{B}_1$, and again let $\varphi(r) = r^{-\beta} + \nu\psi$. Recall that

$$\partial_r \varphi(r) = -\beta r^{-(\beta+1)} + \tfrac{1}{r} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}},$$

and so if we were to choose a constant C_2 large enough it would follow that

$$|\partial_r \varphi(r)| \geq C \tfrac{1}{r} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}},$$

whenever $(x\nu)^{\frac{1}{2}} \geq C_2 r^{-\beta}$, and hence as before

$$|\mathcal{B}_1| \leq C \int\limits_{r^{\beta+1} \geq (|\lambda|\nu)^{-\frac{1}{2}}} r^{-1-\alpha} (x\nu)^{-\frac{2n+1}{4}} dr \leq C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}}.$$

Of course the phase in $\mathcal{B}_2$ is trivially never critical in the complete range, so we obtain the same estimate for $\mathcal{B}_2$ everywhere.

We are thus left with estimating $\mathcal{B}_1$ when $C_1 r^{-\beta} \leq (x\nu)^{\frac{1}{2}} \leq C_2 r^{-\beta}$, which means $r \sim (|\lambda|\nu)^{-\frac{1}{2(\beta+1)}}$. Now making the change of variables $r = s(|\lambda|\nu)^{-\frac{1}{2(\beta+1)}}$ we see that

$$\mathcal{B}_1 = (|\lambda|\nu)^{\frac{\alpha}{2(\beta+1)}} \int_0^\infty \vartheta(s) s^{-1-\alpha} e^{i(|\lambda|\nu)^{\frac{\beta}{2(\beta+1)}} \Phi(s)} \left(\frac{\nu}{x}\right)^{\frac{1}{2}} (x\nu)^{\frac{1-n}{2}} \left(\frac{\psi}{\psi'}\right)^{\frac{1}{2}} \sigma_1(\nu\psi) dr,$$

where ϑ is smooth and supported where $s \sim 1$ and $\Phi(s) = s^{-\beta} + (|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}} \nu\psi$.

Although our phase $\Phi(s)$ may now be stationary in this range we do have the following.

Lemma 4.1. *If $s \sim 1$ and $x \leq \frac{\nu}{2}$, then $|\partial_s \Phi(s)| + |\partial_s^2 \Phi(s)| \geq C_0 > 0$.*

Proof. Recall that

$$\partial_s \Phi(s) = -\beta s^{-(\beta+1)} + (|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}} \tfrac{1}{s} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}},$$

and

$$\partial_s^2 \Phi(s) = \beta(\beta+1) s^{-(\beta+2)} - (|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}} \tfrac{1}{s^2} \tfrac{x^{\frac{3}{2}}}{(\nu-x)^{\frac{1}{2}}}.$$

Hence $\partial_s \Phi(s) = 0$ if and only if

$$(|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}} \tfrac{1}{s} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}} = \beta s^{-(\beta+1)}.$$

It is therefore clear that if

$$(|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}} \tfrac{1}{s} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}} \geq (1 + \tfrac{\beta}{2})\, \beta s^{-(\beta+1)},$$

then

$$|\partial_s \Phi(s)| \geq C s^{-(\beta+1)}.$$

While if

$$(|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}} \tfrac{1}{s} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}} \leq (1 + \tfrac{\beta}{2})\, \beta s^{-(\beta+1)},$$

then we have

$$\partial_s^2\Phi(s) = \beta(\beta+1)s^{-(\beta+2)} - \tfrac{1}{s}\tfrac{x}{\nu-x}(|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}}\tfrac{1}{s}x^{\frac{1}{2}}(\nu-x)^{\frac{1}{2}} \geq Cs^{-(\beta+2)},$$

since $0 < \frac{x}{\nu-x} \leq 1$. $\square$

We note that since

$$-\partial_s^3\Phi(s) = \beta(\beta+1)(\beta+2)s^{-(\beta+3)} + (|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}}\tfrac{\nu}{s^3}\tfrac{x^{\frac{3}{2}}}{(\nu-x)^{\frac{3}{2}}} > 0 \tag{18}$$

our phase Φ can have at most two separated critical points, that they must be separated follows from the fact that for all $\ell = 0, 1, \ldots$ we have $|\partial_s^\ell\Phi(s)| \leq c_\ell$. In addition to this we note that if we now set

$$a(s) = \vartheta(s)s^{-1-\alpha}(x\nu)^{\frac{1-n}{2}}\Big(\frac{\nu}{x}\Big)^{\frac{1}{2}}\Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}}\sigma_1(\nu\psi),$$

then for all $\ell = 0, 1, \ldots$

$$|\partial_s^\ell a(s)| \leq C(x\nu)^{\frac{1-n}{2}-\frac{1}{4}} \sim (|\lambda|\nu)^{-\frac{(n-\frac{1}{2})\beta}{2(\beta+1)}}.$$

Applying van der Corput's lemma therefore gives

$$|\mathcal{B}_1| \leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}}.$$

We have therefore established that $|\mathcal{B}| \leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}}$, and hence the same estimate for $\mathcal{I}_2$.

4.3. **Oscillatory interval II:** $\frac{\nu}{2} \leq x \leq \nu - \nu^{\frac{1}{3}}$**.** We now have $\nu^{\frac{1}{3}} \leq \nu - x \leq \frac{\nu}{2}$ and the trivial estimate $|\Lambda_k^\delta(x)| \leq Cx^{-\frac{1}{4}}(\nu-x)^{-\frac{1}{4}}$.

The situation here is much the same as it was in §4.2 only here we must use instead the Airy asymptotic form. In order to do better than the trivial estimate we shall again make use the oscillation in the main term of our asymptotic expansion. It follows from Lemma 3.2 that

$$\phi \sim \frac{\nu - x}{\nu},$$

from this and (15) it is immediately clear that

$$\frac{1}{10} \leq \phi' \leq 10 \quad \text{and} \quad \phi'' \leq C.$$

Case 1: $(x\nu)^{\frac{1}{2}} \leq C_1 r^{-\beta}$. We should integrate by parts and since $x \leq (x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$I_3 = \int_0^\infty \chi_3^\epsilon(r)r^{-1-\alpha+N\beta}e^{ir^{-\beta}}(x\nu)^{\frac{N+1-n}{2}}\Lambda_{k-N}^{n-1+N}(x)dr.$$

Using the Airy asymptotic forms (14) we may write this as $I_3 = cA + E_A$, where

$$A = \int_0^\infty \chi_3^\epsilon(r)r^{-1-\alpha+N\beta}e^{ir^{-\beta}}(x\nu)^{\frac{N+1-n}{2}}x^{-\frac{1}{2}}\nu^{\frac{1}{6}}\big(\tfrac{1}{-\phi'}\big)^{\frac{1}{2}}\operatorname{Ai}(-\nu^{\frac{2}{3}}\phi)dr.$$

Error term:

$$\begin{aligned}
|E_A| &\le \int_0^\infty \chi_3^\epsilon(r)(r)r^{-1-\alpha+N\beta}(x\nu)^{\frac{N+1-n}{2}}x^{-\frac{3}{2}}\nu^{\frac{1}{6}}\big(\tfrac{1}{-\phi'}\big)^{\frac{1}{2}}|\widetilde{\mathrm{Ai}}(-\nu^{\frac{2}{3}}\phi)|dr \\
&\le C\int_0^\infty \chi_3^\epsilon(r)(r)r^{-1-\alpha+N\beta}(x\nu)^{\frac{1}{2}(N-\frac{1}{2}-n)}\nu^{\frac{1}{4}}(\nu-x)^{-\frac{1}{4}}dr \\
&\le C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}}\nu^{-\frac{3}{4}}\int_{\nu-x\le\nu}(\nu-x)^{-\frac{1}{4}}dx \\
&\le C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}},
\end{aligned}$$

provided N is chosen large enough.

Main term: Recall that for $z>0$

$$\mathrm{Ai}(-z) = \tfrac{1}{3}z^{\frac{1}{2}}[J_{1/3}(\tfrac{2}{3}z^{\frac{3}{2}}) + J_{-1/3}(\tfrac{2}{3}z^{\frac{3}{2}})],$$

and since $J_{1/3}$ and $J_{-1/3}$ satisfy the same bounds for large z it suffices to estimate

$$\widetilde{A} = \int_0^\infty \chi_3^\epsilon(r)r^{-1-\alpha+N\beta}e^{ir^{-\beta}}(x\nu)^{\frac{N+1-n}{2}}x^{-\frac{1}{2}}\nu^{\frac{1}{6}}\big(\tfrac{1}{-\phi'}\big)^{\frac{1}{2}}(\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}}J_{1/3}(\tfrac{2}{3}\nu\phi^{\frac{3}{2}})dr.$$

It follows from (16) that

$$J_{1/3}(\tfrac{2}{3}\nu\phi^{\frac{3}{2}}) = \sigma_1(\nu\phi^{\frac{3}{2}})e^{i\frac{2}{3}\nu\phi^{\frac{3}{2}}} + \sigma_2(\nu\phi^{\frac{3}{2}})e^{-i\frac{2}{3}\nu\phi^{\frac{3}{2}}},$$

where σ_i is a symbol of order $-\frac{1}{2}$. We therefore write $\widetilde{A} = A_1 + A_2$, where

$$\begin{aligned}
A_1 &= \int_0^\infty \chi_3^\epsilon(r)r^{-1-\alpha+N\beta}e^{i[r^{-\beta}+\frac{2}{3}\nu\phi^{\frac{3}{2}}]}(x\nu)^{\frac{N+1-n}{2}}x^{-\frac{1}{2}}\nu^{\frac{1}{6}}\big(\tfrac{1}{-\phi'}\big)^{\frac{1}{2}}(\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}}\sigma_1(\nu\phi^{\frac{3}{2}})dr, \\
A_2 &= \int_0^\infty \chi_3^\epsilon(r)r^{-1-\alpha+N\beta}e^{i[r^{-\beta}-\frac{2}{3}\nu\phi^{\frac{3}{2}}]}(x\nu)^{\frac{N+1-n}{2}}x^{-\frac{1}{2}}\nu^{\frac{1}{6}}\big(\tfrac{1}{-\phi'}\big)^{\frac{1}{2}}(\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}}\sigma_2(\nu\phi^{\frac{3}{2}})dr.
\end{aligned}$$

Let us first consider the integral A_1, and now let $\widetilde{\varphi}(r) = r^{-\beta} + \frac{2}{3}\nu\phi^{\frac{3}{2}}$. We note that $\partial_r\widetilde{\varphi} = \partial_r\varphi$. It therefore follows that $\widetilde{\varphi}$ behaves exactly as φ did in §4.2 and so for C_1 chosen small enough we again may integrate by parts. In this case our amplitude

$$\widetilde{a}(r) = \chi_3^\epsilon(r)r^{-1-\alpha+N\beta}(x\nu)^{\frac{N+1-n}{2}}x^{-\frac{1}{2}}\nu^{\frac{1}{6}}\big(\tfrac{1}{-\phi'}\big)^{\frac{1}{2}}(\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}}\sigma_1(\nu\phi^{\frac{3}{2}}),$$

courtesy of the symbol estimates $|\partial_r^\ell\sigma_1(\nu\phi^{\frac{3}{2}})| \le Cr^{-\ell}x^{\frac{1}{4}+\ell}(\nu-x)^{-\frac{3}{4}-\ell}$, satisfies for $\ell = 0, 1, \dots$ the differential inequality

$$|\partial_r^\ell\widetilde{a}(r)| \le Cr^{-1-\alpha+N\beta-\ell}(\nu-x)^{-\ell-\frac{1}{4}}\nu^{N-n+\frac{3}{4}+\ell}.$$

Integrating by parts N' times we therefore get the estimate

$$\begin{aligned}
|A_1| &\le C\int_0^\infty r^{-1-\alpha+(N+N')\beta}(\nu-x)^{-N'-\frac{1}{4}}\nu^{N+N'-n+\frac{3}{4}}dr \\
&\le C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{4})\beta}{2(\beta+1)}}\int_{\nu^{\frac{1}{3}}\le\nu-x}(\nu-x)^{-N'-\frac{1}{4}}dx \\
&\le C(|\lambda|\nu)^{\frac{\alpha-n\beta-\frac{N'}{3}}{2(\beta+1)}},
\end{aligned}$$

again provided N large enough, but also in this case that $N' \ge 1$.

We of course obtain the same estimate for A_2 since its phase is trivially never stationary.

Case 2: $(x\nu)^{\frac{1}{2}} \geq C_1 r^{-\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\mathcal{I}_3 = \int_0^\infty \chi_3^\epsilon(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} \Lambda_k^{n-1}(x) dr.$$

Using the asymptotic forms (14) we may write $\mathcal{I}_3 = c\mathcal{A} + \mathcal{E}_A$, where

$$\mathcal{A} = \int_0^\infty \chi_3^\epsilon(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\tfrac{1}{-\phi'}\right)^{\frac{1}{2}} \operatorname{Ai}(-\nu^{\frac{2}{3}}\phi) dr.$$

Error term:

$$\begin{aligned}
|\mathcal{E}_A| &\leq \int_0^\infty \chi_3^\epsilon(r)(r) r^{-1-\alpha} (x\nu)^{\frac{1-n}{2}} x^{-\frac{3}{2}} \nu^{\frac{1}{6}} \left(\tfrac{1}{-\phi'}\right)^{\frac{1}{2}} |\widetilde{\operatorname{Ai}}(-\nu^{\frac{2}{3}}\phi)| dr \\
&\leq C \int_0^\infty \chi_3^\epsilon(r)(r) r^{-1-\alpha} (x\nu)^{-\frac{1}{2}(n+\frac{1}{2})} \nu^{\frac{1}{4}} (\nu - x)^{-\frac{1}{4}} dr \\
&\leq C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}} \nu^{-\frac{3}{4}} \int_{\nu - x \leq \nu} (\nu - x)^{-\frac{1}{4}} dx \\
&\leq C(|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{2})\beta}{2(\beta+1)}}.
\end{aligned}$$

Main term: As above it suffices to consider

$$\widetilde{\mathcal{A}} = \int_0^\infty \chi_3^\epsilon(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\tfrac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} J_{1/3}(\tfrac{2}{3}\nu\phi^{\frac{3}{2}}) dr,$$

and as before we shall write $\widetilde{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1 = \int_0^\infty \chi_3^\epsilon(r) r^{-1-\alpha} e^{i[r^{-\beta} + \frac{2}{3}\nu\phi^{\frac{3}{2}}]} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\tfrac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_1(\nu\phi^{\frac{3}{2}}) dr,$$

$$\mathcal{A}_2 = \int_0^\infty \chi_3^\epsilon(r) r^{-1-\alpha} e^{i[r^{-\beta} - \frac{2}{3}\nu\phi^{\frac{3}{2}}]} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\tfrac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_2(\nu\phi^{\frac{3}{2}}) dr.$$

So again matter reduce to the study of $\mathcal{A}_1$ and we again let $\widetilde{\varphi}(r) = r^{-\beta} + \frac{2}{3}\nu\phi^{\frac{3}{2}}$. As in §4.2 we may choose a constant C_2 so large that

$$|\partial_r \widetilde{\varphi}(r)| \geq C \tfrac{1}{r} x^{\frac{1}{2}} (\nu - x)^{\frac{1}{2}}$$

whenever $(x\nu)^{\frac{1}{2}} \geq C_2 r^{-\beta}$. In this range we can therefore integrate by parts N' times and obtain

$$\begin{aligned}
|\mathcal{A}_1| &\leq C \int_0^\infty r^{-1-\alpha} (\nu - x)^{-\frac{3}{2}N' - \frac{1}{4}} \nu^{\frac{1}{2}N' - n + \frac{3}{4}} dr \\
&\leq C(|\lambda|\nu)^{\frac{\alpha - n\beta}{2(\beta+1)}} \nu^{\frac{1}{2}N' - \frac{1}{4}} \int_{\nu^{\frac{1}{3}} \leq \nu - x} (\nu - x)^{-\frac{3}{2}N' - \frac{1}{4}} dx \\
&\leq C(|\lambda|\nu)^{\frac{\alpha - n\beta}{2(\beta+1)}}.
\end{aligned}$$

Of course the phase in $\mathcal{A}_2$ is trivially never critical in the complete range, so we obtain the same estimate for $\mathcal{A}_2$ everywhere.

We are thus left with estimating $\mathcal{A}_1$ when $C_1 r^{-\beta} \leq (x\nu)^{\frac{1}{2}} \leq C_2 r^{-\beta}$, which means $r \sim (|\lambda|\nu)^{-\frac{1}{2(\beta+1)}}$. Making the change of variables $r = s(|\lambda|\nu)^{-\frac{1}{2(\beta+1)}}$ we see that

$$\mathcal{A}_1 = (|\lambda|\nu)^{\frac{\alpha}{2(\beta+1)}} \int_0^\infty \vartheta(s) s^{-1-\alpha} e^{i(|\lambda|\nu)^{\frac{\beta}{2(\beta+1)}} \widetilde{\Phi}(s)} (x\nu)^{\frac{1-n}{2}} x^{-\frac{1}{2}} \nu^{\frac{1}{6}} \left(\tfrac{1}{-\phi'}\right)^{\frac{1}{2}} (\nu^{\frac{2}{3}}\phi)^{\frac{1}{2}} \sigma_1(\nu\phi^{\frac{3}{2}}) dr,$$

where ϑ is smooth and supported where $s \sim 1$ and $\widetilde{\Phi}(s) = s^{-\beta} + (|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}}\nu\psi$.

We recall that $\widetilde{\Phi}(s)$ satisfies the same differential inequalities as $\Phi(s)$ and thus both $\partial_s\widetilde{\Phi}(s)$ and $\partial_s^2\widetilde{\Phi}(s)$ now vanish at a point $s = s_0$ determined by the condition $\frac{x}{\nu-x} = \beta+1$. We can however use the fact, noted before, that

$$-\partial_s^3\Phi(s) = \beta(\beta+1)(\beta+2)s^{-(\beta+3)} + (|\lambda|\nu)^{-\frac{\beta}{2(\beta+1)}}\frac{\nu}{s^3}\frac{x^{\frac{3}{2}}}{(\nu-x)^{\frac{3}{2}}} > 0.$$

Applying the method of stationary phase (see [10], chapter VIII, Proposition 3) in a suitably small neighborhood of s_0 therefore gives

$$\begin{aligned}\mathcal{A}_1 &= C(|\lambda|\nu)^{\frac{\alpha-\frac{1}{3}\beta}{2(\beta+1)}}\nu^{-n+\frac{1}{2}}e^{i(|\lambda|\nu)^{\frac{\beta}{2(\beta+1)}}\widetilde{\Phi}(s_0)} + O\Big((|\lambda|\nu)^{\frac{\alpha-\frac{2}{3}\beta}{2(\beta+1)}}\nu^{-n+\frac{1}{2}}\Big)\\ &= C(|\lambda|\nu)^{\frac{\alpha-(n-\frac{1}{6})\beta}{2(\beta+1)}}e^{i(|\lambda|\nu)^{\frac{\beta}{2(\beta+1)}}\widetilde{\Phi}(s_0)} + O\Big((|\lambda|\nu)^{\frac{\alpha-(n+\frac{1}{6})\beta}{2(\beta+1)}}\Big).\end{aligned}$$

Away from this small neighborhood one has $|\partial_s^2\Phi(s)| + |\partial_s^2\Phi(s)| \geq C_0$ and hence we may argue as in Oscillatory interval I, *Case 2* and obtain the estimate

$$|\mathcal{A}_1| \leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}}.$$

We have therefore established that

$$\mathcal{A} = C(|\lambda|\nu)^{\frac{\alpha-(n-\frac{1}{6})\beta}{2(\beta+1)}}e^{i(|\lambda|\nu)^{\frac{\beta}{2(\beta+1)}}\widetilde{\Phi}(s_0)} + O\Big((|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}}\Big),$$

and hence the same equality for $\mathcal{I}_3$.

4.4. **Neighborhood of the turning point:** $|\nu - x| \leq \nu^{\frac{1}{3}}$**.** Here we just use a size estimate and the fact that $|\Lambda_k^\delta(x)| \leq C\nu^{-\frac{1}{3}}$. This is the best we can do since $\nu\phi^{\frac{3}{2}} \leq \nu\big(\frac{\nu-x}{\nu}\big)^{\frac{3}{2}} \leq 1$.

Case 1: $(x\nu)^{\frac{1}{2}} \leq C_1 r^{-\beta}$. We should integrate by parts and since $x \leq C(x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$\begin{aligned}|I_4| &= \left|\int_0^\infty \chi_4^\epsilon(r)r^{-1-\alpha+N\beta}e^{ir^{-\beta}}(x\nu)^{\frac{N+1-n}{2}}\Lambda_{k-N}^{n-1+N}(x)dr\right|\\ &\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}}\int_{|\nu-x|\leq\nu^{\frac{1}{3}}}\nu^{-\frac{1}{3}}dx\\ &\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}},\end{aligned}$$

provided N is taken large enough.

Case 2: $(x\nu)^{\frac{1}{2}} \geq C_1 r^{-\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\begin{aligned}|\mathcal{I}_4| &= \left|\int_0^\infty \chi_4^\epsilon(r)r^{-1-\alpha}e^{ir^{-\beta}}(x\nu)^{\frac{1-n}{2}}\Lambda_k^{n-1}(x)dr\right|\\ &\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}}\int_{|\nu-x|\leq\nu^{\frac{1}{3}}}\nu^{-\frac{1}{3}}dx\\ &\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}}.\end{aligned}$$

4.5. **Monotonic region I:** $\nu+\nu^{\frac{1}{3}} \leq x \leq \frac{3}{2}\nu$**.** Recall that here $|\Lambda_k^\delta(x)| \leq C\nu^{-\frac{1}{4}}(x-\nu)^{-\frac{1}{4}}e^{-\gamma_1\nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}}$.

Case 1: $(x\nu)^{\frac{1}{2}} \leq C_1 r^{-\beta}$. We should integrate by parts and since $x \leq C(x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$\begin{aligned}
|I_5| &= \left| \int_0^\infty \chi_5^\epsilon(r) r^{-1-\alpha+N\beta} e^{ir^{-\beta}} (x\nu)^{\frac{N+1-n}{2}} \Lambda_{k-N}^{n-1+N}(x)dr \right| \\
&\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}} \int_{x-\nu \geq \nu^{\frac{1}{3}}} \nu^{-\frac{1}{4}} (x-\nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}} dx \\
&\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}} \int_{u\geq 1} u^{-\frac{1}{4}} e^{-\gamma_1 u^{\frac{3}{2}}} du \\
&\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}},
\end{aligned}$$

provided N is chosen large enough.

Case 2: $(x\nu)^{\frac{1}{2}} \geq C_1 r^{-\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\begin{aligned}
|\mathcal{I}_5| &= \left| \int_0^\infty \chi_5^\epsilon(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} \Lambda_k^{n-1}(x)dr \right| \\
&\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}} \int_{x-\nu \geq \nu^{\frac{1}{3}}} \nu^{-\frac{1}{4}} (x-\nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}} dx \\
&\leq C(|\lambda|\nu)^{\frac{\alpha-n\beta}{2(\beta+1)}},
\end{aligned}$$

as before.

4.6. **Monotonic region II:** $x \geq \frac{3\nu}{2}$**.** Here we have the trivial estimate $|\Lambda_k^\delta(x)| \leq Ce^{-\gamma_2 x}$.

Case 1: $x \leq C_1 r^{-\beta}$. We should integrate by parts and since $x \geq (x\nu)^{\frac{1}{2}}$ it suffices to estimate,

$$\begin{aligned}
|I_6| &= \left| \int_0^\infty \chi_6^\epsilon(r) r^{-1-\alpha+N\beta} e^{ir^{-\beta}} x^N (x\nu)^{\frac{1-n}{2}} |\Lambda_k^{n-1}(x)| dr \right| \\
&\leq C|\lambda|^{\frac{\alpha-N\beta}{\beta+2}} \nu^{1-n} \int_{x\geq\nu} x^{N-1} e^{-\gamma_2 x} dx \\
&\leq C|\lambda|^{\frac{\alpha-N\beta}{\beta+2}} \nu^{N-n} e^{-\gamma_2 \nu},
\end{aligned}$$

for all N large enough.

Case 2: $x \geq C_1 r^{-\beta}$. Here we shall not integrate by parts first, so we wish to estimate

$$\begin{aligned}
|\mathcal{I}_6| &= \left| \int_0^\infty \chi_6^\epsilon(r) r^{-1-\alpha} e^{ir^{-\beta}} (x\nu)^{\frac{1-n}{2}} \Lambda_k^{n-1}(x) dr \right| \\
&\leq C|\lambda|^{\frac{\alpha-N\beta}{\beta+2}} \nu^{1-n} \int_{x\geq\nu} x^{N-1} e^{-\gamma_2 x} dx \\
&\leq C|\lambda|^{\frac{\alpha-N\beta}{\beta+2}} \nu^{N-n} e^{-\gamma_2 \nu},
\end{aligned}$$

for all $N \geq 0$.

This completes the proof of Theorem 2.1.

5. Proof of Theorem 1.3

5.1. **Establishing Sufficiency.** We have already noted above that Theorem 1.1 is essentially no more than a special case of Theorem 1.3, the case $\lambda = 0$. The sufficiency in Theorem 1.3 is however, as we shall see, an almost immediate consequence of that in Theorem 1.1 and it is to this that we now turn our attention.

5.1.1. *Establishing Sufficiency in Theorem 1.1.* We may clearly assume that our kernel $K_{\alpha,\beta}(x, y)$ is supported in a small neighborhood of the diagonal, as in the complement of such a region $K_{\alpha,\beta}$ is, for $\alpha > 0$, clearly dominated by an integrable function of $|x - y|$.

In order to establish the positive result we shall dyadically decompose the operator

$$T = \sum_{j=0}^{\infty} T_j.$$

In order to do this we consider the following partition of unity; choose $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ such that $\sum_{j=0}^{\infty} \vartheta(2^j r) = 1$ for all $0 \leq r \leq 1$, and then write for $f \in \mathcal{S}$ with compact support

$$T_j f(x) = \int K_j(x, y) f(y)\, dy$$

whenever x is in the complement of the support of f, where

$$K_j(x, y) = \vartheta(2^j|x - y|) K_{\alpha,\beta}(x, y).$$

The key result here is the following.

Theorem 5.1. *The operator norms of T_j are uniformly bounded whenever $\alpha \leq \frac{d\beta}{2}$, more precisely*

$$\int_{\mathbf{R}^d} |T_j f(x)|^2 dx \leq C 2^{j(2\alpha - d\beta)} \int_{\mathbf{R}^d} |f(x)|^2 dx.$$

We note that as the operator norms of T_j are equal to that of

$$\widetilde{T}_j f(x) = 2^{j\alpha} \int_{\mathbf{R}^d} \vartheta(|x - y|) a_{2^{-j}}(x, y) e^{i 2^{j\beta} \varphi_{2^{-j}}(x,y)} f(y) dy$$

it suffices to consider the operators $\widetilde{T}_j$ and establish the following inequality

$$\int_{|x - x_0| \leq 1} |\widetilde{T}_j f(x)|^2 dx \leq C 2^{j(2\alpha - d\beta)} \int_{|x - x_0| \leq 10} |f(x)|^2 dx, \tag{19}$$

uniformly for all x_0 in $\mathbf{R}^d$. Integrating (19) with respect to x_0 then gives Lemma 5.1.

Key to establishing (19) is the following proposition of Hörmander, which may be thought of as a variable coefficient version of Plancherel's theorem. For a proof see [10], Chapter IX.

Proposition 5.2. *Let Ψ be a smooth function of compact support in x and y and Φ be real-valued and smooth. If we assume that,*

$$\det\Big(\frac{\partial^2 \Phi}{\partial x_i \partial y_j}\Big) \neq 0,$$

on the support of Ψ, *then*

$$\Big\|\int_{\mathbf{R}^d}\Psi(x,y)e^{i\lambda\Phi(x,y)}f(y)dy\Big\|_{L^2(\mathbf{R}^d)}\leq C\lambda^{-\frac{d}{2}}\|f\|_{L^2(\mathbf{R}^d)}.$$

Proof of (19). We shall first assume that $x_0=0$ and write $f=f_1+f_2$, with f_1 is supported in $B(10)$, f_2 supported outside $B(9)$, f_1 and f_2 smooth, and with $|f_1|,|f_2|\leq|f|$. We fix $\chi\in C_0^\infty$ so that $\chi\equiv 1$ in $B(1)$.

Now, since the integral kernel of $\widetilde{T}_j$ is compactly supported in $x-y$, it follows that $\chi\widetilde{T}_j$ is compactly supported in x and y, so applying proposition 5.2 we see that,

$$\begin{aligned}\int_{B(1)}|\widetilde{T}_jf_1(x)|dx&=\int_{\mathbf{R}^d}|\chi(x)\widetilde{T}_jf_1(x)|dx\\&\leq C2^{j(2\alpha-d\beta)}\int_{\mathbf{R}^d}|f_1(x)|^2dx\\&\leq C2^{j(2\alpha-d\beta)}\int_{B(10)}|f(x)|^2dx.\end{aligned}$$

However, if $|x|\leq 1$, and $|y|\geq 9$, it follows that $|x-y|\geq 8$ and hence $\chi\widetilde{T}_jf_2\equiv 0$.

The passage to general x_0 is then easy since although $\widetilde{T}_j$ are not translation invarient the 'translated' kernels

$$K^{x_0}_{\alpha,\beta}(x,y)=K_{\alpha,\beta}(x+x_0,y+x_0)$$

do satisfy assumptions (2) and (3), for the same α and β as $K_{\alpha,\beta}$, uniformly in x_0, and hence

$$\begin{aligned}\int_{|x-x_0|\leq 1}|\widetilde{T}_jf(x)|^2dx&=\int_{|x-x_0|\leq 1}\left|2^{-jd}\int K_j(2^{-j}x,2^{-j}y)f(y)\,dy\right|^2dx\\&=\int_{|x|\leq 1}\left|2^{-jd}\int K_j^{x_0}(2^{-j}x,2^{-j}y)f(y+x_0)dy\right|^2dx\\&\leq C2^{j(2\alpha-d\beta)}\int_{|x|\leq 10}|f(x+x_0)|^2dx\\&\leq C2^{j(2\alpha-d\beta)}\int_{|x-x_0|\leq 10}|f(x)|^2dx,\end{aligned}$$

where $K_j^{x_0}(x,y)=\vartheta(2^j(x-y))K^{x_0}_{\alpha,\beta}(x,y)$. □

Theorem 1.1 now follows from Lemma 5.1 and an application of Cotlar's lemma (plus a standard limiting argument) once we have verified that the T_j are, in the following sense, almost orthogonal.

Lemma 5.3. *If* $\alpha=\frac{d\beta}{2}$ *then* $\|T_i^*T_j\|_{Op}+\|T_iT_j^*\|_{Op}\leq C2^{-\frac{d\beta}{2}|i-j|}$.

Proof. This follows trivially from Lemma 5.1 whenever $|i-j|\leq 10$, since $\|T_i^*T_j\|_{Op}\leq\|T_i\|_{Op}\|T_j\|_{Op}$. We shall therefore, without loss of generality, assume that $j\geq i+10$. Now $T_i^*T_j$ has a kernel

$$L_{ij}(x,y)=\int\bar{K}_i(z,x)K_j(z,y)dz,$$

and the same operator norm as the operator with kernel

$$\begin{aligned}\widetilde{L}_{ij}(x,y) &= 2^{-jd}L_{ij}(2^{-j}x,2^{-j}y)\\ &= 2^{-jd}\int \bar{K}_i(z,2^{-j}x)K_j(z,2^{-j}y)dz\\ &= 2^{j2\alpha}\int\limits_{\substack{|z-y|\sim 1\\ |z-x|\sim 2^{j-i}}} a_{2^{-j}}(z,x)a_{2^{-j}}(z,y)e^{i2^{j\beta}[\varphi_{2^{-j}}(z,y)-\varphi_{2^{-j}}(z,x)]}dz.\end{aligned}$$

Trivially we get the estimate $|\widetilde{L}_{ij}(x,y)| \le C2^{j2\alpha}2^{(i-j)(d+\alpha)}$. However from (2c) it follows that

$$|\nabla_z[\varphi_{2^{-j}}(z,y)-\varphi_{2^{-j}}(z,x)]| \ge C_0,$$

thus there is always a direction in which we may integrating by parts, in doing so d times we obtain

$$|\widetilde{L}_{ij}(x,y)| \le C2^{j(2\alpha-d\beta)}2^{(i-j)(d+\alpha)} = 2^{(i-j)(d+\alpha)}.$$

This of course implies that

$$\sup_x \int |\widetilde{L}_{ij}(x,y)|\,dy \le C2^{(i-j)\alpha}$$

and

$$\sup_y \int |\widetilde{L}_{ij}(x,y)|\,dx \le C2^{(i-j)\alpha},$$

and so by Schur's Lemma we are done. □

5.1.2. *Establishing Sufficiency in Theorem 1.3.* The task of proving the positive half of Theorem 1.3 reduces, as above, to establishing the following result.

Theorem 5.4. *The inequality*

$$\int_{|z-z_0|\le 1} |T^\lambda f(z)|^2 dz \le C_\lambda \int_{|z-z_0|\le 10} |f(z)|^2 dz,$$

holds with constant C_λ independent of z_0.

As before Theorem 1.3 then follows immediately from Theorem 5.4 via an integration in z_0.

Proof. The case when $\lambda = 0$ of course follows form estimate (19) above, so in what follows it is understood that $\lambda \neq 0$. Using the fact that for $z, w \in \mathbf{C}^n$ we have

$$\left| e^{i\frac{1}{2}\lambda\,\mathrm{Im}\, z\cdot\bar{w}} - \sum_{k=0}^{N-1}\frac{\lambda^k}{4^k k!}\left(\overline{(z-w)}\cdot w - (z-w)\cdot\overline{w}\right)^k \right| \le C_\lambda |w|^N|z-w|^N,$$

one sees that matters reduce to estimating operators of the form

$$f \mapsto \int_{\mathbf{C}^n} \mathcal{K}(z,w)f(w)dw,$$

where

$$\mathcal{K}(z,w) = K_{\alpha,\beta}(z,w)(\bar{z}-\bar{w})^\ell(z-w)^m$$

for $|\ell| + |m| = k = 0,\dots,N-1$.

It is possible to then establish that these operators are bounded in $L^2(\mathbf{C}^n)$ whenever $\alpha - k \le n\beta$ in the model case by appealing to spherical harmonics and Fourier transform methods. It is however easy to see that these kernels $\mathcal{K}(z,w)$ are in fact strongly singular integral kernels and it therefore follows immediately from Theorem 1.1, in particular inequality (19), that these more general operators are also bounded in $L^2(\mathbf{C}^n)$ whenever $\alpha - k \leq n\beta$, this establishes Theorem 5.4 and hence the sufficiency of $\alpha \leq n\beta$ in Theorem 1.3. □

5.2. **Establishing Necessity.** We begin by noting that the model kernels $K_{\alpha,\beta}(x,y) = \widetilde{K}_{\alpha,\beta}(x-y)$ are indeed *strongly singular integral kernels.* That they satisfy the differential inequalities (2) is self evident and the simple lemma below establishes that they in addition also satisfy the required non-degeneracy hypothesis.

Lemma 5.5. *Let $\varphi(x,y) = |x-y|^{-\beta}$, then $\det\big(\frac{\partial^2\varphi}{\partial x_i \partial y_j}\big) \neq 0$ whenever $\beta \neq -1$.*

Proof. Recall that $\nabla|x|^{-\beta} = -\beta|x|^{-\beta-1}\frac{x}{|x|}$, it is then easy to see that

$$\partial_{x_i}\partial_{y_j}\varphi(x,y) = \beta|x-y|^{-\beta-2}\left(\delta_{ij} - (\beta+2)u_i u_j\right),$$

where $u_i = \frac{(x-y)_i}{|x-y|}$. We therefore need to check that $I - (\beta+2)uu^t$ is non-singular. To do this we shall denote by R the rotation matrix such that $Ru = e_1$, of course $\det R = 1$ and it is clear that

$$\det(I - (\beta+2)uu^t) = \det\big(R(I - (\beta+2)uu^t)R^t\big) = 1 - (\beta+2) = -(\beta+1). \qquad \square$$

In order to establish the necessity of the condition $\alpha \leq n\beta$ it therefore suffices to test our model twisted convolution operator R^λ on a suitably chosen $L^2(\mathbf{C}^n)$ function f_0 over an appropriately chosen range of z. We choose as our test function $f_0(z) = |z|^{-\gamma}\chi(10|z|)$, where $\gamma < n$. Restricting ourselves to small $|z|$ we therefore have

$$\chi(10|z|)R^\lambda f_0(z) = \chi(10|z|)\int_{\mathbf{C}^n} |z-w|^{-2n-\alpha} e^{i(|z-w|^{-\beta} + \frac{1}{2}\lambda \operatorname{Im} z\cdot\bar{w})}\chi(10|w|)|w|^{-\gamma}dw.$$

At this point we also make the observation that Sf_0 is a radial function and as such we may with no loss in generality assume that $z = (|z|, 0, \ldots, 0)$.

Now making the change of variables $w = |z|s$ we see that

$$\chi(10|z|)Sf_0(z) = \chi(10|z|)|z|^{-\alpha-\gamma}\int_{\mathbf{C}^n} e^{i|z|^{-\beta}\varphi(s,|z|)}\psi(s)\,ds,$$

where

$$\varphi(s,|z|) = (1 - 2s_1 + |s|^2)^{-\frac{\beta}{2}} + \frac{1}{2}\lambda|z|^{\beta+2}s_2$$

and

$$\psi(s) = (1 - 2s_1 + |s|^2)^{-n-\frac{\alpha}{2}}\chi(10|s|)|s|^{-\gamma}.$$

We have therefore now reduced matters to the analysis of the oscillatory integral

$$I(|z|) = \int_{\mathbf{C}^n} e^{i|z|^{-\beta}\varphi(s,|z|)}\psi(s)\,ds,$$

as $|z| \to 0$. We now write

$$I(|z|) = M(|z|) + E_1(|z|) + E_2(|z|) + E_3(|z|),$$

where

$$M(|z|) = e^{i|z|^{-\beta}}\int_{\mathbf{C}^n} \chi(|z|^{-\beta(1-\epsilon)}|s|)e^{i[c|z|^{-\beta}s_1 + \frac{1}{2}\lambda|z|^2 s_2]}|s|^{-\gamma}\,ds,$$

$$E_1(|z|) = \int_{\mathbf{C}^n} \chi(|z|^{-\beta(1-\epsilon)}|s|)e^{i\frac{1}{2}\lambda|z|^2 s_2}\left[e^{i|z|^{-\beta}(1-2s_1+|s|^2)^{-\frac{\beta}{2}}} - e^{i|z|^{-\beta}(1+cs_1)}\right]|s|^{-\gamma}ds,$$

$$E_2(|z|) = \int_{\mathbf{C}^n} \chi(|z|^{-\beta(1-\epsilon)}|s|)e^{i|z|^{-\beta}\varphi(s,|z|)}\left[\psi(s) - |s|^{-\gamma}\right] ds,$$

and

$$E_3(|z|) = \int_{\mathbf{C}^n} \left[1 - \chi(|z|^{-\beta(1-\epsilon)}|s|)\right]e^{i|z|^{-\beta}\varphi(s,|z|)}\psi(s)\, ds.$$

Let us first take care of the error terms. It is easy to verify that whenever $|s| \leq |z|^{\beta(1-\epsilon)}$ we have

$$\left|e^{i|z|^{-\beta}(1-2s_1+|s|^2)^{-\frac{\beta}{2}}} - e^{i|z|^{-\beta}(1+cs_1)}\right| \leq C|z|^{-\beta\epsilon}|s|$$

and

$$|\psi(s) - |s|^{-\gamma}| \leq C|z|^{\beta(1-\epsilon)}|s|^{-\gamma},$$

and hence that

$$\begin{aligned}|E_1(|z|)| &\leq C|z|^{-\beta\epsilon}\int_{|s|\leq|z|^{\beta(1-\epsilon)}} |s|^{-\gamma+1}\, ds\\ &\leq C|z|^{-\beta\epsilon}|z|^{\beta(2n-\gamma+1)(1-\epsilon)}\\ &= C|z|^{\beta(2n-\gamma)}|z|^{\beta(1-\epsilon(2+2n-\gamma))},\end{aligned}$$

while

$$\begin{aligned}|E_2(|z|)| &\leq C|z|^{\beta(1-\epsilon)}\int_{|s|\leq|z|^{\beta(1-\epsilon)}} |s|^{-\gamma}\, ds\\ &\leq C|z|^{\beta(1-\epsilon)}|z|^{\beta(2n-\gamma)(1-\epsilon)}\\ &= C|z|^{\beta(2n-\gamma)}|z|^{\beta(1-\epsilon(1+2n-\gamma))}.\end{aligned}$$

In the error integral $E_3(|z|)$ it shall be advantageous to repeatedly apply integration by parts in the s_1 direction since $C|z|^{\beta(1-\epsilon)} \leq |s| \leq \frac{1}{10}$. In fact it is clear that in this region

$$\partial_1\varphi(s,|z|) = \beta(1-s_1)(1-2s_1+|s|^2)^{-\frac{\beta+2}{2}} \geq C(\beta),$$

while $|\partial_1^\ell\varphi(s,|z|)| \leq c_\ell$, for all $\ell \geq 0$ and

$$\left|\partial_1^\ell[1-\chi(|z|^{-\beta(1-\epsilon)}|s|)]\psi(s)\right| \leq c_\ell\left(|z|^{-\beta(1-\epsilon)\ell}|s|^{-\gamma}\vartheta(10|z|^{-\beta(1-\epsilon)}|s|) + |s|^{-\gamma-\ell}\right).$$

It therefore follows that after integrating by parts N times we obtain the estimate

$$\begin{aligned}|E_3(|z|)| &\leq C|z|^{\beta N}\left(|z|^{\beta(1-\epsilon)N}\int_{|s|\approx|z|^{\beta(1-\epsilon)}} |s|^{-\gamma}\, ds + \int_{|s|\geq|z|^{\beta(1-\epsilon)}} |s|^{-\gamma-N}\, ds\right)\\ &\leq C|z|^{\beta(2n-\gamma)}|z|^{\beta\epsilon(N-2n+\gamma)}.\end{aligned}$$

It remains for us to show that for $|z|$ small

$$|M(|z|)| \geq C|z|^{\beta(2n-\gamma)}. \tag{20}$$

Assuming this for the moment we see that it would then follow that $|I(|z|)| \geq |z|^{\beta(2n-\gamma)}$ for small enough $|z|$ and hence that

$$\|R^\lambda f_0\|_2^2 \geq C\int_{\mathbf{C}^n} \chi(10|z|)|z|^{-2(\alpha+\gamma-\beta(2n-\gamma))}\, dz.$$

It then follows that if R^λ were to extend to a bounded operator on $L^2(\mathbf{C}^n)$ we must have
$$\alpha - n\beta < (n-\gamma)(\beta+1)$$
for all $\gamma < n$. It follows immediately that one must then necessarily have the condition $\alpha \le n\beta$.

The lower bound estimate (20) for the main term will be an almost immediate consequence of the following, slightly more general lemma.

Lemma 5.6. *If $\gamma < \frac{d}{2}$ then*
$$\int_{\mathbf{R}^d} \chi(|s|)e^{i\xi\cdot s}|s|^{-\gamma}\, ds = C|\xi|^{\gamma-d} + O(|\xi|^{\gamma-d-1}).$$

Let us assume Lemma 5.6 for the moment and see how this gives us (20). To do this we first rescale the integral $M(|z|)$ so that
$$M(|z|) = |z|^{(2n-\gamma)\beta(1-\epsilon)}e^{i|z|^{-\beta}}\int_{\mathbf{C}^n} \chi(|s|)e^{i[c|z|^{-\beta\epsilon}s_1+\frac{1}{2}\lambda|z|^{2+\beta(1-\epsilon)}s_2]}|s|^{-\gamma}\, ds,$$
applying Lemma 5.6 to this then gives
$$M(|z|) = |z|^{(2n-\gamma)\beta}e^{i|z|^{-\beta}}\big(1+(\tfrac{1}{2}\lambda|z|^{\beta+2})^2\big)^{\frac{\gamma-2n}{2}} + O\big(|z|^{(2n-\gamma+\epsilon)\beta}\big(1+(\tfrac{1}{2}\lambda|z|^{\beta+2})^2\big)^{\frac{\gamma-2n-1}{2}}\big).$$
The result now follows if we restrict ourselves at the beginning to $|z| \le \min\{\frac{1}{100}, \lambda^{-\frac{1}{\beta+2}}\}$.

Proof of Lemma 5.6. This is merely a Fourier transform and hence
$$\int_{\mathbf{R}^d} \chi(|s|)e^{i\xi\cdot s}|s|^{-\gamma}\, ds = C\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)|\eta|^{\gamma-d}\, d\eta.$$
Now since χ is smooth and of compact support $\widehat{\chi}$ is a Schwartz function and satisfies the inequality
$$|\widehat{\chi}(|\eta-\xi|)| \le C_N(1+|\eta-\xi|)^{-N},$$
for all $N \ge 0$. Using this standard estimate it is easy to see that whenever $|\xi| \notin [\frac{1}{2}|\eta|, 2|\eta|]$ we have
$$\left|\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)|\eta|^{\gamma-d}\, d\eta\right| \le C|\xi|^{-N+\gamma}.$$
Now if $|\xi| \in [\frac{1}{2}|\eta|, 2|\eta|]$, then
$$\begin{aligned}\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)|\eta|^{\gamma-d}\, d\eta &= |\xi|^{\gamma-d}\int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)\, d\eta + \int_{\mathbf{R}^d} \widehat{\chi}(|\eta-\xi|)\big[|\eta|^{\gamma-d} - |\xi|^{\gamma-d}\big]\, d\eta \\ &= |\xi|^{\gamma-d}\chi(0) + O\Big(\int_{|\eta|\approx|\xi|} |\eta|^{\gamma-d-1}\, d\eta\Big) \\ &= |\xi|^{\gamma-d} + O(|\xi|^{\gamma-d-1}).\end{aligned}$$
□

6. Final Remarks

6.1. **Extensions.** Using similar methods one can also establish Theorem 1.2 for the integral operators
$$R_\gamma f(z,t) = \int_{\mathbf{H}^n} M_\gamma([w,s]^{-1}\cdot[z,t])f(w,s)\, dw\, ds,$$
where
$$M_\gamma(z,t) = \widetilde{K}_{\alpha,\beta}(z)\delta_0(t-|z|^\gamma),$$

with $\gamma \geq 2$, and Theorem 1.3 for the integral operators

$$T_\gamma^\lambda f(z) = \int_{\mathbf{C}^n} K_{\alpha,\beta}(z,w) e^{i\lambda|z-w|^\gamma} e^{i\frac{1}{2}\lambda \operatorname{Im} z\cdot\bar{w}} f(w)dw,$$

with $\gamma \geq 0$.

We plan to discuss these operators in more detail, in particular the uniform behavior of the operators T_γ^λ, is a future paper.

6.2. **Strongly singular Radon transforms on $\mathbf{R}^{d+1}$.** We should finish by saying something about the analogous class of strongly singular Radon transforms on $\mathbf{R}^{d+1}$ formally given by

$$\mathcal{R}_\gamma f = f * L,$$

where $L(x,t) = \widetilde{K}_{\alpha,\beta}(x)\delta_0(t-|x|^\gamma)$ and again $\gamma \geq 0$. The following result is due to Chandarana [2]; see also Zielinski [15].

Theorem 6.1. *If $\gamma \geq 2$, then $\mathcal{R}_\gamma$ extends to a bounded operator on $L^2(\mathbf{R}^2)$ if and only if $\alpha \leq (\frac{1}{2}-\frac{1}{6})\beta$.*

We take this opportunity to remark that the situation when $d \geq 2$ is quite different.

Theorem 6.2. *If $d \geq 2$ and $\gamma \geq 0$, then $\mathcal{R}_\gamma$ extends to a bounded operator on $L^2(\mathbf{R}^{d+1})$ if and only if $\alpha \leq \frac{1}{2}\beta$.*

As with the model operators discussed earlier proving L^2-boundedness is equivalent to establishing the *uniform* boundedness, in $\mathbf{R}^{d+1}$, of the multiplier

$$m(\xi,\lambda) = (2\pi)^{\frac{d}{2}} \int_0^\infty \chi(r) r^{-1-\alpha} e^{i(r^{-\beta}+\lambda r^\gamma)} J_{\frac{d-2}{2}}(r|\xi|)(r|\xi|)^{\frac{2-d}{2}}\, dr.$$

It is then easy to see that for $r|\xi|$ small and λ large the phase in this integral may be stationary and that one can, in this region, only establish the estimate

$$|m(\xi,\lambda)| \leq C|\lambda|^{\alpha-\frac{1}{2}\beta}.$$

When $d \geq 2$ this is in fact the worst region and one can, by applying the method of stationary phase, show that for ξ fixed

$$m(\xi,\lambda) \approx |\lambda|^{\alpha-\frac{1}{2}\beta}.$$

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