ASYMPTOTIC PROPERTIES OF LAGUERRE FUNCTIONS

NEIL LYALL

Recall that Laguerre functions of type δ, $\delta > -1$, form an orthonormal basis for $L^2(\mathbf{R}^+)$ and are given by

$$\Lambda_k^\delta(x) = \big(\tfrac{k!}{(k+\delta)!}\big)^{1/2} L_k^\delta(x) e^{-\frac{1}{2}x} x^{\frac{\delta}{2}},$$

where $L_k^\delta(x) = \sum_{j=0}^k \binom{k+\delta}{k-j} \frac{(-x)^j}{j!}$ are the Laguerre polynomials of type δ.

The two asymptotic formulae below which hold uniformly in their respective ranges of validity (which overlap) are due to Erdélyi [2]; see also [1] and [3]. In what follows $\nu = 4k + 2\delta + 2$ and $N = \nu/4$.

1. The Bessel asymptotic forms

Let $0 \leq x \leq b\nu$, $b < 1$. Then for $k \geq k_0$,

$$\Lambda_k^\delta(x) = \Big(\frac{(\delta+k)!}{k!}\Big)^{\frac{1}{2}} 2^{\delta - \frac{1}{2}} \nu^{-\frac{\delta}{2}} \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} \{J_\delta(\nu\psi) + O[\nu^{-1}(\tfrac{x}{\nu - x})^{\frac{1}{2}} \widetilde{J}_\delta(\nu\psi)]\},$$

and so

$$\Lambda_k^\delta(x) = C_1(\delta) \Big(\frac{\nu}{x}\Big)^{\frac{1}{2}} \Big(\frac{\psi}{\psi'}\Big)^{\frac{1}{2}} \{J_\delta(\nu\psi) + O[\nu^{-1}(\tfrac{x}{\nu - x})^{\frac{1}{2}} \widetilde{J}_\delta(\nu\psi)]\} \tag{1}$$

where $C_1(\delta)$ is a constant independent of k, $\psi = \psi(t)$ satisfies

$$\psi'(t) = \frac{1}{2}\Big(\frac{1}{t} - 1\Big)^{\frac{1}{2}} \tag{2}$$

and $t = \frac{x}{\nu}$. For $0 \leq t < 1$,

$$\psi(t) = \tfrac{1}{2}[(t - t^2)^{\frac{1}{2}} + \sin^{-1} t^{\frac{1}{2}}],$$

and

$$\widetilde{J}_\delta(u) = \begin{cases} J_\delta(u) & \text{if } u \text{ sufficiently small,} \\ \big(|J_\delta(u)|^2 + |Y_\delta(u)|^2\big)^{\frac{1}{2}} & \text{otherwise,} \end{cases}$$

here Y_δ and J_δ are Bessel functions of order δ.

Lemma 1. *If $0 \leq t \leq \frac{1}{2}$, then $\frac{1}{2}t^{\frac{1}{2}} \leq \psi(t) \leq t^{\frac{1}{2}}$.*

Proof. Let $f(t) = (t - t^2)^{\frac{1}{2}} + \sin^{-1} t^{\frac{1}{2}}$, notice then that $f'(t) = \big(\frac{1-t}{t}\big)^{\frac{1}{2}}$. Now if $0 \leq s \leq \frac{1}{2}$, we have $\frac{1}{2}s^{-\frac{1}{2}} \leq f'(s) \leq s^{-\frac{1}{2}}$, and so

$$\tfrac{1}{2}\int_0^t s^{-\frac{1}{2}} ds \leq \int_0^t f'(s) ds \leq \int_0^t s^{-\frac{1}{2}} ds$$

which implies $t^{\frac{1}{2}} \leq f(t) \leq 2t^{\frac{1}{2}}$, since $f(0) = 0$. □

2. The Airy asymptotic forms

Let $0 < a\nu \leq x$, $a > 0$. Then for $k \geq k_0$,

$$\Lambda_k^\delta(x) = \frac{(-1)^k}{\big(k!(\delta+k)!\big)^{\frac{1}{2}}} 2^{\frac{5}{6}} N^{N + \frac{1}{6}} e^{-N} x^{-\frac{1}{2}} \Big(\frac{\pi}{-\phi'}\Big)^{\frac{1}{2}} \{Ai(-\nu^{\frac{2}{3}}\phi) + O[x^{-1}\widetilde{Ai}(-\nu^{\frac{2}{3}}\phi)]\},$$

and so, using Stirling's formula

$$\Lambda_k^\delta(x) = C_2(\delta)(-1)^k \nu^{\frac{1}{6}} x^{-\frac{1}{2}} \Big(\frac{1}{-\phi'}\Big)^{\frac{1}{2}} \{Ai(-\nu^{\frac{2}{3}}\phi) + O[x^{-1}\widetilde{Ai}(-\nu^{\frac{2}{3}}\phi)]\} \tag{3}$$

where $C_2(\delta)$ is a constant independent of k, $\phi = \phi(t)$ satisfies

$$\phi(t)[\phi'(t)]^2 = \frac{1}{4}\Big(\frac{1}{t} - 1\Big), \tag{4}$$

and again $t = \frac{x}{\nu}$. Now one can show

$$\phi(t) = \Big(\frac{3}{4}\Big)^{\frac{2}{3}} \begin{cases} [\cos^{-1} t^{\frac{1}{2}} - (t-t^2)^{\frac{1}{2}}]^{\frac{2}{3}} & \text{if } 0 < t \leq 1, \\ -[(t^2-t)^{\frac{1}{2}} - \cosh^{-1} t^{\frac{1}{2}}]^{\frac{2}{3}} & \text{if } t > 1, \end{cases}$$

and

$$\widetilde{Ai}(z) = \begin{cases} Ai(z) & \text{if } z \geq 0, \\ \big(|Ai(z)|^2 + |Bi(z)|^2\big)^{\frac{1}{2}} & \text{if } z \leq 0, \end{cases}$$

here Ai and Bi are Airy integrals[1].

Lemma 2. *If $\frac{1}{2} \leq t \leq 1$, then $\frac{1}{2}(1-t) \leq \phi(t) \leq 1-t$.*

Proof. Let $g(t) = \cos^{-1} t^{\frac{1}{2}} - (t-t^2)^{\frac{1}{2}}$, notice then that $g'(t) = -\big(\frac{1-t}{t}\big)^{\frac{1}{2}}$. Now if $\frac{1}{2} \leq s \leq 1$, we have $(1-s)^{\frac{1}{2}} \leq -g'(s) \leq 2(1-s)^{\frac{1}{2}}$, and so

$$\int_t^1 (1-s)^{\frac{1}{2}} ds \leq -\int_t^1 g'(s) ds \leq 2\int_t^1 (1-s)^{\frac{1}{2}} ds$$

which implies $\frac{2}{3}(1-t)^{\frac{3}{2}} \leq g(t) \leq \frac{4}{3}(1-t)^{\frac{3}{2}}$, since $g(1) = 0$. ☐

Note also that, for $z > 0$

$$Ai(-z) = \tfrac{1}{3} z^{\frac{1}{2}} [J_{1/3}(\tfrac{2}{3} z^{\frac{3}{2}}) + J_{-1/3}(\tfrac{2}{3} z^{\frac{3}{2}})]$$
$$Bi(-z) = \big(\tfrac{z}{3}\big)^{\frac{1}{2}} [J_{1/3}(\tfrac{2}{3} z^{\frac{3}{2}}) + J_{-1/3}(\tfrac{2}{3} z^{\frac{3}{2}})].$$

3. Trivial Estimates

It follows from the asymptotics above that for k large we have the following crude estimates for our Laguerre function; see Askey and Wainger [1].

$$|\Lambda_k^\delta(x)| \leq C \begin{cases} (x\nu)^{\frac{\delta}{2}} & \text{if } 0 \leq x \leq \frac{1}{\nu}, \\ (x\nu)^{-\frac{1}{4}} & \text{if } \frac{1}{\nu} \leq x \leq \frac{\nu}{2}, \\ \nu^{-\frac{1}{4}}(\nu - x)^{-\frac{1}{4}} & \text{if } \frac{\nu}{2} \leq x \leq \nu - \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{3}} & \text{if } \nu - \nu^{\frac{1}{3}} \leq x \leq \nu + \nu^{\frac{1}{3}}, \\ \nu^{-\frac{1}{4}}(x-\nu)^{-\frac{1}{4}} e^{-\gamma_1 \nu^{-\frac{1}{2}}(x-\nu)^{\frac{3}{2}}} & \text{if } \nu + \nu^{\frac{1}{3}} \leq x \leq \frac{3\nu}{2}, \\ e^{-\gamma_2 x} & \text{if } x \geq \frac{3\nu}{2}, \end{cases}$$

where $\gamma_1, \gamma_2 > 0$ are fixed constants.

References

[1] R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, American J. Math., 87 (1965), pp. 695–708.

[2] A. Erdélyi, *Asymptotic forms for Laguerre polynomials*, J. Indian Math. Soc., 24 (1960), pp. 235–250.

[3] C. L. Frenzen and R. Wong, *Uniform asymptotic expansions of Laguerre polynomials*, SIAM J. Math. Anal., 19 (1988), pp. 1232–1248.

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[1] Recall that $Ai(z)$ and $Bi(z)$ are independent solutions of the differential equation $\frac{d^2y}{dz^2} = zy$ and have the integral representations $Ai(z) = \dfrac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3 + zt)dt$ and $Bi(z) = \dfrac{1}{\pi}\int_0^\infty \big\{e^{\frac{1}{3}t^3+zt} + \sin(\frac{1}{3}t^3 + zt)\big\}dt$.