STRONGLY SINGULAR INTEGRAL OPERATORS ASSOCIATED TO DIFFERENT QUASI-NORMS ON THE HEISENBERG GROUP

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ABSTRACT. In this article we study the behavior of strongly singular integrals associated to three different, albeit equivalent, quasi-norms on Heisenberg groups; these quasi-norms give rise to phase functions whose mixed Hessians may or may not drop rank along suitable varieties. In the particular case of the Koranyi norm we improve on the arguments in [7] and obtain sharp L^2 estimates for the associated operators.

1. INTRODUCTION

The Heisenberg group $\mathbf{H}_a^n$ is a non-commutative nilpotent Lie group, with underlying manifold $\mathbf{R}^{2n+1}$ equipped the group law

$$(1) \qquad (x,t) \cdot (y,s) = (x+y, s+t - 2a\, x^{\mathrm{t}} J y)$$

where a is a nonzero real number and J denotes the standard symplectic matrix on $\mathbf{R}^{2n}$, namely

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

with inverses given by $(x,t)^{-1} = -(x,t)$. The *nonisotropic* dilations

$$(2) \qquad (x,t) \mapsto (\delta x, \delta^2 t).$$

are automorphisms of $\mathbf{H}_a^n$ and as such the homogeneous dimension of this group is $2n+2$.

We will consider here, for different quasi-norms $\rho(x,t)$ on $\mathbf{H}_a^n$, the class of model (group) convolution operators formally given by

$$Tf(x,t) = f * K_{\alpha,\beta}(x,t)$$

where $K_{\alpha,\beta}$ is a strongly singular distributional kernel on $\mathbf{H}_a^n$ that agrees, for $(x,t) \neq (0,0)$, with the function

$$K_{\alpha,\beta}(x,t) = \rho(x,t)^{-2n-2-\alpha} e^{i\rho(x,t)^{-\beta}} \chi(\rho(x,t)),$$

where $\beta > 0$ and χ is smooth and compactly supported in a small neighborhood of the origin.

Operators of this type were first studied, using Fourier transform techniques, in the Euclidean setting of $\mathbf{R}^d$ with $\rho(x) = |x|$ by Hirschman [4] in the case $d = 1$ and then in higher dimensions by Wainger [10], C. Fefferman [2], and C. Fefferman and Stein [3]. For some generalizations and an oscillatory integral approach to these classical results, see Lyall [6]. The analogous questions on the Heisenberg group were first investigated by the second author in [7].

1.1. **Statement of main results.** There are various natural choices of quasi-norm on $\mathbf{H}_a^n$, for example one can take

$$\rho_0(x,t) = \max\{|x_1|, \ldots, |x_{2n}|, |t|^{1/2}\},$$

however this choice is not smooth away from the origin. We shall instead consider the following three equivalent quasi-norms which clearly are smooth away from the origin;

(i) $\rho_1(x,t) = (|x|^4 + t^2)^{1/4}$

(ii) ρ_2 defined by $\rho_2(x,t) = 1 \Longleftrightarrow |x|^2 + t^2 = 1$ and extended by homogeneity

(iii) $\rho_3(x,t) = (x_1^4 + \cdots + x_{2n}^4 + t^2)^{1/4}$.

The case when $\rho(x,t) = \rho_1(x,t)$, the so called Koranyi norm on the Heisenberg group, was initially studied by the second author in [7] using the group Fourier transform.

Theorem (Lyall [7]). *Let $\rho(x,t) = \rho_1(x,t)$ and $a \neq 0$.*

(i) *If $\alpha \leq n\beta$, then T extends to a bounded operator from $L^2(\mathbf{H}_a^n)$ to itself.*
(ii) *If T extends to a bounded operator from $L^2(\mathbf{H}_a^n)$ to itself, then $\alpha \leq (n + \frac{1}{2})\beta$.*

The proof of this result relied upon the radial nature of the Koranyi norm and uniform asymptotic expansions for Laguerre functions due to Erdélyi [1] along with some careful analysis.

In this article we optimally sharpen the previously obtained estimates and in addition address the question of the behavior of different quasi-norms for the first time. We do not employ the Fourier transform in our arguments and as such our methods are not restricted to the class of translation-invariant operators.

We fix the constant

$$C_\beta = \frac{\beta + 2}{2}\left(2\beta + 5 + \sqrt{(2\beta + 5)^2 - 9}\right)$$

and note that $C_\beta \geq 9$ for all $\beta > 0$. Our first main result is then the following.

Theorem 1. *If $\rho(x,t) = \rho_1(bx, bt)$ with $0 < a^2/b^2 < C_\beta$, then T extends to a bounded operator from $L^2(\mathbf{H}_a^n)$ to itself if and only if $\alpha \leq (n + \frac{1}{2})\beta$.*

The analogous result for $\rho(x,t) = \rho_2(x,t)$, the nonisotropic Minkowski functional associated to the (Euclidean) unit ball, is the following.

Theorem 2. *If $\rho(x,t) = \rho_2(bx, bt)$ with $a^2/b^2 \leq 1$, then T extends to a bounded operator from $L^2(\mathbf{H}_a^n)$ to itself whenever $\alpha \leq (n + \frac{1}{2})\beta$.*

We note that the Koranyi norm $\rho_1(x,t)$ is indeed an actual norm on $\mathbf{H}_a^n$ for all $0 < a^2 \leq 1$, as is the nonisotropic Minkowski functional $\rho_2(x,t)$ for sufficiently small $|a|$.

A negative result for the quasi-norm $\rho_3(x,t)$ that relates to the proofs of Theorems 1 and 2 is discussed in Section 5.

2. REDUCTIONS AND REMARKS

2.1. **Reduction to dyadic estimates.** The necessary condition in Theorem 1 follows from the arguments in [7]. To establish sufficiency in both Theorems 1 and 2 matters reduce to considering the dyadic operators

$$T_j(x,t) = f * K_j(x,t),$$

where

$$K_j(x,t) = \vartheta(2^j \rho(x,t)) K_{\alpha,\beta}(x,t),$$

with $\vartheta \in C_0^\infty(\mathbf{R})$ supported in $[\frac{1}{2}, 2]$ is chosen such that $\sum_{j=0}^\infty \vartheta(2^j r) = 1$ for all $0 \le r \le 1$.

As in [7] everything reduces to establishing the following key dyadic estimates.

Theorem 3. *If $\alpha \le (n + \frac{1}{2})\beta$ and either*

   (i) $\rho(x,t) = \rho_1(bx, bt)$ *with* $0 < a^2/b^2 < C_\beta$, *or*
   (ii) $\rho(x,t) = \rho_2(bx, bt)$ *with* $a^2/b^2 \le 1$,

then the dyadic operators T_j are bounded uniformly on $L^2(\mathbf{H}_a^n)$, more precisely

$$(3) \qquad \int_{\mathbf{H}_a^n} |T_j f(x,t)|^2 \, dx \, dt \le C 2^{j(2\alpha - (2n+1)\beta)} \int_{\mathbf{H}_a^n} |f(x,t)|^2 \, dx \, dt.$$

Theorems 1 and 2 then follow from an application of Cotlar's lemma (and a standard limiting argument) since the operators T_j are, in the following sense, almost orthogonal.

Proposition 4. *If $\alpha \le (n + \frac{1}{2})\beta$, then*

$$\|T_j^* T_{j'}\|_{L^2(\mathbf{H}_a^n) \to L^2(\mathbf{H}_a^n)} + \|T_j T_{j'}^*\|_{L^2(\mathbf{H}_a^n) \to L^2(\mathbf{H}_a^n)} \le C 2^{-(n+\frac{1}{2})\beta|j-j'|}.$$

This follows exactly as in [7] once we have made the observation that if $\rho(x,t)$ is any quasi-norm on $\mathbf{H}_a^n$ satisfying the estimate $c^{-1} \le \rho(\delta x, \delta^2 t) \le c$ for some $c \ge 1$ and $\delta > 0$, then there exists a constant $c_0 > 1$ so that either

$$c_0^{-1} \le \frac{\partial}{\partial x_j} \rho(x,t) \le c_0$$

for some $j = 1, \ldots, 2n$, or

$$c_0^{-1} \delta \le \frac{\partial}{\partial t} \rho(x,t) \le c_0 \delta.$$

Since from this it follows that

$$\left| \nabla_{(y,s)} [\rho(y,s)^{-\beta} - \rho((x,t) \cdot (y,s))^{-\beta}] \right| \ge C > 0,$$

whenever $\rho((x,t) \cdot (y,s)) \gg \rho(y,s)$. For more details see [7].

2.2. **Remarks on $L^p \to L^q$ estimates.** For $p \le q$, the $L^p \to L^q$ operator norms of T_j are equal to that of the rescaled operator $\widetilde{T}_j$ given by

$$\widetilde{T}_j f(x,t) = 2^{jh(1/p-1/q)} 2^{j\alpha} S_j f(x,t)$$

where $h = 2n + 2$ denotes the homogeneous dimension of $\mathbf{H}_a^n$ and

$$S_j f(x,t) = \int_{\mathbf{H}_a^n} \widetilde{K}_j\big((y,s)^{-1} \cdot (x,t)\big) f(y,s) \, dy \, ds$$

with

$$\widetilde{K}_j(x,t) = \vartheta(\rho(x,t))\rho(x,t)^{-h-\alpha}e^{i2^{j\beta}\rho(x,t)^{-\beta}}.$$

In particular establishing Theorem 3 is equivalent to showing that the L^2 operator norm of S_j is $O(2^{-jd\beta/2})$, where $d = 2n+1$ denotes the topological dimension of $\mathbf{H}_a^n$, standard interpolation techniques then give that the L^p operator norm of S_j is $O(2^{-jd\beta/2}2^{jd\beta|1/p-1/2|})$. If $p \leq 2$ this is also a bound for the $L^p \to L^{p'}$, $L^p \to L^2$, and $L^{p'} \to L^2$ operator norms and can, for this range of p, be written more succinctly as $O(2^{-jd\beta/p'})$.

From the corresponding results for the dyadic operator T_j, which of course follow immediately from those for S_j, we obtain the following.

Theorem 5. *If $1 < p < \infty$ and either*

    (i) $\rho(x,t) = \rho_1(bx, bt)$ *with* $0 < a^2/b^2 < C_\beta$, *or*
    (ii) $\rho(x,t) = \rho_2(bx, bt)$ *with* $a^2/b^2 \leq 1$,

then T extends to a bounded operator from $L^p(\mathbf{H}_a^n)$ to itself whenever

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{d\beta - 2\alpha}{2d\beta}.$$

The estimates above match those obtained by Wainger [10] for the corresponding operators in $\mathbf{R}^d$. However, unlike the situation in the Euclidean setting which was settled in [2] and [3], the behavior of the operator T on the Heisenberg group near L^1 and the endpoint results in L^p for $p \neq 2$ remain open problems.

We chose to use the notation h and d for the homogeneous and topological dimensions of $\mathbf{H}_a^n$ in the arguments alluded to above as these also apply in the setting of homogeneous groups; of course establishing the analogue of Theorem 3 in this more general group settings remains an open problem.

3. Homogeneous groups and a proposition of Hörmander

The Heisenberg group is of course one of the simplest examples of a (non-commutative) homogeneous group. Recall that a homogeneous group consists of $\mathbf{R}^d$ equipped with a Lie group structure, together with a family of dilations

$$x = (x_1, \ldots, x_d) \mapsto \delta \circ x = (\delta^{a_1}x_1, \ldots, \delta^{a_d}x_d),$$

with $a_1, \ldots, a_d$ strictly positive, that are group automorphisms, for all $\delta > 0$.

To each homogeneous group on $\mathbf{R}^d$, we can associate its Lie algebra, consisting of left-invariant vector fields on $\mathbf{R}^d$, with basis $\{X_j\}_{1 \leq j \leq d}$ where each X_j is the left-invariant vector field that agrees with $\partial/\partial x_j$ at the origin.

Key to establishing Theorem 3 is the following, presumably well known, generalization of a proposition of Hörmander [5], see also [9], Chapter IX.

Proposition 6. *Let Ψ be a smooth function of compact support in x and y, and Φ be real-valued and smooth on the support of Ψ. If we assume that*

$$(4) \qquad\qquad \det\left(X_j Y_k \Phi(x,y)\right) \neq 0,$$

on the support of Ψ, then for $\lambda > 0$ we have

$$(5) \qquad \left\| \int_{\mathbf{R}^d} \Psi(x,y) e^{i\lambda \Phi(x,y)} f(y) dy \right\|_{L^2(\mathbf{R}^d)} \leq C\lambda^{-\frac{d}{2}} \|f\|_{L^2(\mathbf{R}^d)}.$$

Proposition 6 can in fact be extended to families of smooth vector fields $X_1, \ldots, X_d$ forming a basis at every point of $\mathbf{R}^d$; however, we choose to state it in this restricted generality (which is already more than we need) as this admits a proof which is simply the natural modification of Hörmander's original argument.

Proof. By using a partition of unity we may assume that the amplitude Ψ has suitably small compact support in both x and y. Denoting the operator on the left hand side of inequality (5) by T_λ it is then easy to see that

$$T_\lambda^* T_\lambda f(y) = \int_{\mathbf{R}^d} K_\lambda(x,z) f(z) \, dz$$

where

$$K_\lambda(x,z) = \int_{\mathbf{R}^d} e^{i\lambda[\Phi(x,y) - \Phi(z,y)]} \Psi(x,y) \overline{\Psi(z,y)} \, dy.$$

It therefore suffices to establish the kernel estimate

$$(6) \qquad |K_\lambda(x,z)| \leq C(1 + \lambda|z^{-1} \cdot x|)^{-N},$$

since from this it would follow that

$$\int |K_\lambda(x,z)| \, dz \approx |\{z : |z^{-1} \cdot x| \leq \lambda^{-1}\}| = C\lambda^{-d}$$

and similarly for $\int |K_\lambda(x,z)| \, dx$, and therefore by Schur's test that

$$\|T_\lambda^* T_\lambda f\|_{L^2(\mathbf{R}^d)} \leq C\lambda^{-d} \|f\|_{L^2(\mathbf{R}^d)}.$$

The kernel $K_\lambda(x,z)$ is of course always bounded, hence in order to establish (6) we need only consider the case when $|z^{-1} \cdot x| \geq \lambda^{-1}$. Now

$$Y_k \Phi(x,y) - Y_k \Phi(z,y) = \int_0^1 \frac{d}{dt} Y_k \Phi(z \cdot t(z^{-1} \cdot x), y) \, dt$$

$$= \sum_{j=1}^d (z^{-1} \cdot x)_j \int_0^1 X_j Y_k \Phi(z \cdot t(z^{-1} \cdot x), y) \, dt$$

$$= \sum_{j=1}^d (z^{-1} \cdot x)_j X_j Y_k \Phi(x,y) + O(|z^{-1} \cdot x|^2).$$

So if we let

$$A = A(x,y) = X_j Y_k \Phi(x,y) \qquad \text{and} \qquad u = u(x,y,z) = A^{-1} \frac{z^{-1} \cdot x}{|z^{-1} \cdot x|}$$

and define

$$\Delta(x,y,z) = (u_1 Y_1 + \cdots + u_d Y_d)[\Phi(x,y) - \Phi(z,y)]$$

it follows that

$$\Delta(x,y,z) = |z^{-1} \cdot x| + O(|z^{-1} \cdot x|^2).$$

Therefore for $|z^{-1} \cdot x|$ small enough, it is here that we use our initial suitably small support assumption, we have

$$|\Delta(x,y,z)| \geq \tfrac{1}{2}|z^{-1} \cdot x|,$$

and if we now set

$$D = \frac{1}{i\lambda\Delta(x,y,z)}(u_1 Y_1 + \cdots + u_d Y_d),$$

it follows that

$$\left| \int_{\mathbf{R}^d} e^{i\lambda[\Phi(x,y) - \Phi(z,y)]} \Psi(x,y)\overline{\Psi(z,y)} \, dy \right| = \left| \int_{\mathbf{R}^d} D^N \left(e^{i\lambda[\Phi(x,y) - \Phi(z,y)]} \right) \Psi(x,y)\overline{\Psi(z,y)} \, dy \right|$$

$$= \left| \int_{\mathbf{R}^d} e^{i\lambda[\Phi(x,y) - \Phi(z,y)]} (D^{\mathrm{t}})^N \left(\Psi(x,y)\overline{\Psi(z,y)} \right) dy \right|$$

$$\leq C_N (1 + \lambda|z^{-1} \cdot x|)^{-N},$$

for all $N \geq 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Proof of Theorem 3

Since the operator norms of T_j are equal to that of the rescaled operator $\widetilde{T}_j$ given by

$$\widetilde{T}_j f(x,t) = 2^{j\alpha} \int_{\mathbf{H}_a^n} \widetilde{K}_j\big((y,s)^{-1} \cdot (x,t)\big) f(y,s) \, dy \, ds$$

where

$$\widetilde{K}_j(x,t) = \vartheta(\rho(x,t))\rho(x,t)^{-2n-2-\alpha} e^{i2^{j\beta}\rho(x,t)^{-\beta}}$$

it suffices to establish estimate (3) for the rescaled operators $\widetilde{T}_j$.

Since the $\widetilde{T}_j$ are local operators, in the sense that the support of $\widetilde{T}_j f$ is always contained in a fixed dilate of some nonisotropic ball containing the support of f, we may make the additional assumption that the integral kernels above have compact support in both (x,t) and (y,s). Estimate (3) for $\widetilde{T}_j$ then follows from Proposition 6 once we have verified the *non-degeneracy* condition (4) in this setting.

It is well known that

$$X_j^\ell = \frac{\partial}{\partial x_j} + 2ax_{j+n}\frac{\partial}{\partial t}, \quad X_{j+n}^\ell = \frac{\partial}{\partial x_{j+n}} - 2ax_j\frac{\partial}{\partial t} \qquad j = 1, \ldots, n,$$

and $T = \frac{\partial}{\partial t}$ form a real basis for the Lie algebra of left-invariant vector fields on $\mathbf{H}_a^n$, while

$$X_j^r = \frac{\partial}{\partial x_j} - 2ax_{j+n}\frac{\partial}{\partial t}, \quad X_{j+n}^r = \frac{\partial}{\partial x_{j+n}} + 2ax_j\frac{\partial}{\partial t} \qquad j = 1, \ldots, n,$$

and $T = \frac{\partial}{\partial t}$ form a real basis for the Lie algebra of right-invariant vector fields.

For convenience we shall use synonymously $X_{2n+1}^\ell = X_{2n+1}^r = T$, and furthermore denote

$$X^\ell = (X_1^\ell, \ldots, X_{2n+1}^\ell) \quad \text{and} \quad X^r = (X_1^r, \ldots, X_{2n+1}^r).$$

We note that

$$-[X_j^r \widetilde{\varphi}](x,t) = [X_j^\ell \varphi]\big((x,t)^{-1}\big),$$

where $\widetilde{\varphi}(x) = \varphi\big((x,t)^{-1}\big)$, and hence

$$X_j^\ell Y_k^\ell \left[\varphi\big((y,s)^{-1} \cdot (x,t)\big) \right] = -[X_j^\ell X_k^r \varphi]\big((y,s)^{-1} \cdot (x,t)\big).$$

The *non-degeneracy* condition (4) in this setting is therefore equivalent to the following.

Proposition 7. *Let* $\Phi(x,t) = \rho(x,t)^{-\beta}$ *with* $\beta > 0$. *If* $(x,t) \neq (0,0)$ *and either*

    (i) $\rho(x,t) = \rho_1(x,t)$ *with* $0 < a^2 < C_\beta$, *or*
    (ii) $\rho(x,t) = \rho_2(x,t)$ *with* $a^2 \leq 1$,

then

$$\det\left(X_j^\ell X_k^r \Phi(x,t)\right) \neq 0.$$

Theorem 3 now follows immediately for $b = 1$, the proof in general follows from the observation that $\mathbf{H}_a^n$ is isomorphic to $\mathbf{H}_{a/b}^n$ with the explicit isomorphism being given by

$$\phi(x,t) = (bx, bt).$$

5. The determinant calculations

The purpose of this section is to prove Proposition 7, however we shall start by stating and sketching the proof of a related negative result for the quasi-norm $\rho_3(x,t)$ on $\mathbf{H}_a^1$. Outlining this argument first will be instructive as it is simpler than, while still similar to, those for Proposition 7.

Proposition 8. *Let* $n = 1$ *and* $\Phi(x,t) = \rho_3(x,t)^{-\beta}$, *then*

$$\det\left(X_j^\ell X_k^r \Phi(x,t)\right) = 0$$

along the lines $(0, x_2, 0)$ *and* $(x_1, 0, 0)$.

Proof. Let $\varphi_3(x,t) = \rho_3(x,t)^4 = x_1^4 + x_2^4 + t^2$. It is straightforward to see that the 'mixed' Hessian of Φ is given by

$$X_j^\ell X_k^r \Phi(x,t) = -\tfrac{\beta}{4}\varphi_3^{-(\beta+8)/4}\{\varphi_3 X_j^\ell X_k^r \varphi_3 - \tfrac{\beta+4}{4} X_j^\ell \varphi_3 X_k^r \varphi_3\}.$$

For convenience both here and in the proofs of both parts of Proposition 7 we define

$$A := X_j^\ell X_k^r \varphi_3 \qquad \text{and} \qquad B := X_j^\ell \varphi_3 X_k^r \varphi_3.$$

Now since rank$(B) = 1$ it follows that

$$\det(\varphi_3 A - \tfrac{\beta+4}{4} B) = \varphi_3^2 \left\{ \varphi_3 \det(A) - \frac{\beta+4}{4} \left\{ \det\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} + \det\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_2 \\ \mathbf{a}_3 \end{pmatrix} + \det\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{b}_3 \end{pmatrix} \right\} \right\},$$

where $\mathbf{a}_j = (a_{j1}, a_{j2}, a_{j3})$ and $\mathbf{b}_j = (b_{j1}, b_{j2}, b_{j3})$.

It is easy to verify that

$$X_1^\ell \varphi_3(x,t) = 4(x_1^3 + ax_2x_3), \quad X_2^\ell \varphi_3(x,t) = 4(x_2^3 - ax_1x_3), \quad \text{and} \quad X_3^\ell \varphi_3(x,t) = 2t,$$

while

$$X_1^r \varphi_3(x,t) = 4(x_1^3 - ax_2x_3), \quad X_2^r \varphi_3(x,t) = 4(x_2^3 + ax_1x_3), \quad \text{and} \quad X_3^r \varphi_3(x,t) = 2t,$$

and hence that

$$A = 2(C + aD)$$

where

$$C = \begin{pmatrix} 6x_1^2 & 2t & 0 \\ -2t & 6x_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -4ax_2^2 & 4ax_1x_2 & 2x_2 \\ 4ax_1x_2 & -4ax_1^2 & -2x_1 \\ -2x_2 & 2x_1 & 1/a \end{pmatrix}$$

Since $\mathrm{rank}(D) = 1$ it follows that

$$\det(A) = 8 \det \begin{pmatrix} 6x_1^2 & 2t \\ -2t & 6x_2^2 \end{pmatrix} = 32(9x_1^2x_2^2 + t^2).$$

For the first of the remaining three determinants we note that

$$\det \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = 8X_1^\ell \varphi_3 \det(E + aF),$$

where

$$E = \begin{pmatrix} 2x_1^3 & 2x_2^3 & 0 \\ -2t & 6x_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} -2x_2t & 2x_1t & t/a \\ 4ax_1x_2 & -4ax_1^2 & -2x_1 \\ -2x_2 & 2x_1 & 1/a \end{pmatrix}.$$

Using the fact that $\mathrm{rank}(F) = 1$ we then see that

$$\det \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = 8X_1^\ell \varphi_3 \det \begin{pmatrix} 2x_1^3 & 2x_2^3 \\ -2t & 6x_2^2 \end{pmatrix}.$$

In an almost identical manner we can also obtain that

$$\det \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_2 \\ \mathbf{a}_3 \end{pmatrix} = 8X_2^\ell \varphi_3 \det \begin{pmatrix} 6x_1^2 & 2t \\ 2x_1^3 & 2x_2^3 \end{pmatrix},$$

and

$$\det \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{b}_3 \end{pmatrix} = 8X_3^\ell \varphi_3 \left\{ 2ax_2 \det \begin{pmatrix} -2t & 6x_2^2 \\ 2x_1^3 & 2x_2^3 \end{pmatrix} + 2ax_1 \det \begin{pmatrix} 6x_1^2 & 2t \\ 2x_1^3 & 2x_2^3 \end{pmatrix} + t \det \begin{pmatrix} 6x_1^2 & 2t \\ -2t & 6x_2^2 \end{pmatrix} \right\},$$

we leave the details to the reader. Bringing all of this together we get that

$$\det(\varphi_3 A - \tfrac{\beta+4}{4} B) = -16\varphi_3^2 \left\{ 6(\beta+1)\varphi_3 x_1^2 x_2^2 + (\beta+2)t^4 + 3(\beta+4)x_1^2 x_2^2 t^2 - 2(x_1^4 + x_2^4)t^2 \right\}.$$

$\square$

5.1. Proof of Proposition 7, part (i).

Let $\varphi_1(x,t) = \rho_1(x,t)^4 = |x|^4 + t^2$. It is straightforward to see that the 'mixed' Hessian of Φ is given by

$$X_j^\ell X_k^r \Phi(x,t) = -\tfrac{\beta}{4}\varphi_1^{-(\beta+8)/4} \{ \varphi_1 X_j^\ell X_k^r \varphi_1 - \tfrac{\beta+4}{4} X_j^\ell \varphi_1 X_k^r \varphi_1 \}.$$

We again define $A := X_j^\ell X_k^r \varphi_1$ and $B := X_j^\ell \varphi_1 X_k^r \varphi_1$. Since $\mathrm{rank}(B) = 1$ it follows that

$$\det(\varphi_1 A - \tfrac{\beta+4}{4} B) = \varphi_1^{2n} \left\{ \varphi_1 \det(A) - \frac{\beta+4}{4} \sum_{j=1}^{2n+1} \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n+1} \end{pmatrix} \right\},$$

where $\mathbf{a}_j = (a_{j1}, \ldots, a_{j\,2n+1})$ and $\mathbf{b}_j = (b_{j1}, \ldots, b_{j\,2n+1})$.

It is an easy calculation to see that

$$X^\ell \varphi_1(x,t) = \left(4|x|^2 x + 4at(Jx), 2t\right),$$

and

$$X^r \varphi_1(x,t) = \left(4|x|^2 x - 4at(Jx), 2t\right),$$

where J is the standard symplectic matrix on $\mathbf{R}^{2n}$ coming from the group structure. Hence we have

$$A = 4 \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + 8 \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} + 4aE \quad \text{and} \quad B = 4|x|^2 \left(F \ \ 0\right) + 4atG$$

where

$$C = |x|^2 I + atJ, \quad D = xx^{\mathrm{t}}, \quad E = \begin{pmatrix} 2a(Jx)(x^{\mathrm{t}}J) & Jx \\ x^{\mathrm{t}}J & 1/2a \end{pmatrix},$$

$$F = (X^\ell \varphi_1)x^{\mathrm{t}}, \quad \text{and} \quad G = \left((X^\ell \varphi_1)(x^{\mathrm{t}}J) \ \ X^\ell \varphi_1/2a\right).$$

Now since both $\mathrm{rank}(D) = 1$ and $\mathrm{rank}(E) = 1$ it follows that

$$\det(A) = 2^{4n+1} \det(C + 2D)$$

$$= 2^{4n+1} \left\{ \left(|x|^4 + a^2 t^2\right)^n + \tfrac{1}{2}\left(|x|^4 + a^2 t^2\right)^{n-1} \sum_{j=1}^{2n} x_j X_j^\ell \varphi_1 \right\}$$

$$= 2^{4n+1} \left(|x|^4 + a^2 t^2\right)^{n-1} \left(3|x|^4 + a^2 t^2\right).$$

To obtain the final identity above we used the fact that

$$\sum_{j=1}^{2n} x_j X_j^\ell \varphi_1 = 4|x|^4.$$

We note that for all j for which it makes sense, both

(7)
$$\mathrm{rank} \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{g}_j \\ \vdots \\ \mathbf{e}_{2n+1} \end{pmatrix} = 1 \quad \text{and} \quad \mathrm{rank} \begin{pmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{f}_j \\ \vdots \\ \mathbf{d}_{2n} \end{pmatrix} = 1.$$

From this and the observation that

$$\sum_{j=1}^{2n} (X_j^\ell \varphi_1)^2 = 16|x|^2 \left(|x|^4 + a^2 t^2\right)$$

we may therefore conclude that

$$\sum_{j=1}^{2n} \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n+1} \end{pmatrix} = 2^{4n+1} \sum_{j=1}^{2n} \det \left\{ \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix} + \begin{pmatrix} 2\mathbf{d}_1 \\ \vdots \\ |x|^2\mathbf{f}_j \\ \vdots \\ 2\mathbf{d}_{2n} \end{pmatrix} \right\}$$

$$= 2^{4n+1}|x|^2 \sum_{j=1}^{2n} X_j^\ell \varphi_1 \det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j + x^{\mathrm{t}} \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix}$$

$$= 2^{4n-1}\left(|x|^4 + a^2t^2\right)^{n-1}|x|^2 \sum_{j=1}^{2n} (X_j^\ell \varphi_1)^2$$

$$= 2^{4n+3}|x|^4 \left(|x|^4 + a^2t^2\right)^n.$$

Finally, we can combine (7) and the fact that

$$\det(C + 2D) = \left(|x|^4 + a^2t^2\right)^{n-1}\left(3|x|^4 + a^2t^2\right),$$

together with the identity

$$\sum_{j=1}^n (x_j X_{j+n}^\ell \varphi_1 - x_{j+n} X_j^\ell \varphi_1) = -4a|x|^2 t$$

to obtain

$$\det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{2n} \\ \mathbf{b}_{2n+1} \end{pmatrix} = 4^{2n+1} a\, X_{2n+1}^\ell \varphi_1 \det \begin{pmatrix} C + 2D & Jx \\ |x|^2 x^{\mathrm{t}} & t/2a \end{pmatrix}$$

$$= 4^{2n+1} t \left\{ 2a|x|^2 \sum_{j=1}^n \left\{ x_j \det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_{j+n} + x^{\mathrm{t}} \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix} - x_{j+n} \det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j + x^{\mathrm{t}} \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix} \right\} + t \det(C + 2D) \right\}$$

$$= 4^{2n+1} t\left(|x|^4 + a^2t^2\right)^{n-1}\left\{ \tfrac{1}{2}a|x|^2 \sum_{j=1}^n (x_j X_{j+n}^\ell \varphi_1 - x_{j+n} X_j^\ell \varphi_1) + t\left(3|x|^4 + a^2t^2\right) \right\}$$

$$= 4^{2n+1} t^2 \left(|x|^4 + a^2t^2\right)^{n-1}\left((3 - 2a^2)|x|^4 + a^2t^2\right).$$

Bringing this all together we see that

$$\sum_{j=1}^{2n+1} \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n+1} \end{pmatrix} = 4^{2n+1} \left(|x|^4 + a^2 t^2\right)^{n-1} \left\{2|x|^8 + t^2\left(3|x|^4 + a^2 t^2\right)\right\},$$

and consequently

$$\det(\varphi_1 A - \tfrac{\beta+4}{4} B) = -(4\varphi_1)^{2n} \left(|x|^4 + a^2 t^2\right)^{n-1} f_1(x,t)$$

where

$$f_1(x,t) = 2(\beta+1)|x|^8 + (\beta+2)t^2\left(3|x|^4 + a^2 t^2\right) - 2|x|^4 a^2 t^2$$
$$= 2(\beta+1)|x|^8 + \left(3(\beta+2) - 2a^2\right)|x|^4 t^2 + (\beta+2)a^2 t^4.$$

By analyzing the discriminant

$$\Delta = 4a^4 - 4(\beta+2)(2\beta+5)a^2 + 9(\beta+2)^2,$$

we see that our Hessian will be *non-degenerate* provided either

$$2a^2 \le 3(\beta+2) \qquad \text{or} \qquad |2a^2 - (2\beta+5)(\beta+2)| < (\beta+2)\sqrt{(2\beta+5)^2 - 9},$$

which reduces simply to the condition that

$$a^2 < C_\beta = \frac{\beta+2}{2}\left(2\beta+5 + \sqrt{(2\beta+5)^2 - 9}\right).$$

Remark 9. We conclude by remarking that when $a^2 \ge C_\beta$ the Hessian degenerates along the paraboloids

$$|x|^4 = \frac{2a^2 - 3(\beta+2) \pm \sqrt{\Delta}}{4(\beta+1)} t^2.$$

In particular when $a^2 = C_\beta$ we have that $\Delta = 0$ and hence the Hessian degenerates along the paraboloid

$$|x|^4 = \frac{2C_\beta - 3(\beta+2)}{4(\beta+1)} t^2 = \frac{(\beta+1)(\beta+2) + \sqrt{(\beta+1)(\beta+4)}}{2(\beta+1)} t^2.$$

5.2. **Proof of Proposition 7, part (ii).** We start by letting $\varphi_2(x,t) = \rho_2(x,t)^2$ and noting that as a consequence φ_2 must satisy the identity

$$(8) \qquad \varphi_2^{-1}|x|^2 + \varphi_2^{-2}t^2 = 1.$$

Arguing as in the proof of Proposition 7, part (i) (and using the same notation) we see that φ_2 satisfies

$$\det(\varphi_2 A - \tfrac{\beta+2}{2} B) = \varphi_2^{2n} \left\{ \varphi_2 \det(A) - \frac{\beta+2}{2} \sum_{j=1}^{2n+1} \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n+1} \end{pmatrix} \right\}.$$

It is an easy calculation to see that

$$\mathcal{A}X^\ell \varphi_2(x,t) = \left(2\varphi_2^{-1}x + 4at(Jx), 2\varphi_2^{-2}t\right),$$

and

$$\mathcal{A}X^r \varphi_2(x,t) = \left(2\varphi_2^{-1}x - 4at(Jx), 2\varphi_2^{-2}t\right),$$

where J is the standard symplectic matrix on $\mathbf{R}^{2n}$ coming from the group structure and

(9) $$\mathcal{A} = \varphi_2^{-2}|x|^2 + 2\varphi_2^{-3}t^2 = \varphi_2^{-1} + \varphi_2^{-3}t^2.$$

A further (somewhat lengthy) calculation then gives that

$$\mathcal{A}^2 A = 2\mathcal{A}\varphi_2^{-2} \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + 2\mathcal{A}\varphi_2^{-4}D,$$

where

$$C = \varphi_2 I + 2atJ$$

and

$$D = [tX^\ell \varphi_2 - \varphi_2(2aJx, 1)][tX^r \varphi_2 - \varphi_2(-2aJx, 1)]^{\mathsf{t}}.$$

Now since $\mathrm{rank}(D) = 1$ it follows that

(10) $$\det(\mathcal{A}^2 A) = 2^{2n+1}\mathcal{A}^{2n-1}\varphi_2^{-(4n+6)}(\varphi_2^2 + 4a^2t^2)^n|x|^4,$$

here we used the fact that

$$\det(C) = (\varphi_2^2 + 4a^2t^2)^n$$

and

$$\mathcal{A}(tX_{2n+1}^\ell \varphi_2 - \varphi_2) = \mathcal{A}(tX_{2n+1}^r \varphi_2 - \varphi_2) = -\varphi_2^{-1}|x|^2.$$

Using the fact that for all $j = 1, \ldots, 2n+1$

(11) $$\mathrm{rank}\begin{pmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{0}_j \\ \vdots \\ \mathbf{d}_{2n+1} \end{pmatrix} = 1$$

together with the observation that

(12) $$\det\begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j + 2\varphi_2^{-1}x^{\mathsf{t}} \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix} = \varphi_2^{-1}(\varphi_2^2 + 4a^2t^2)^{n-1}\mathcal{A}X_j^\ell \varphi_2$$

and some simple reductions we may conclude that

$$\mathcal{A}^{4n+2} \sum_{j=1}^{2n} \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n+1} \end{pmatrix} = 4^n \mathcal{A}^{2n} \varphi_2^{-(4n+2)} |x|^2 \sum_{j=1}^{2n} X_j^\ell \varphi_2 \det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j + 2\varphi_2^{-1} x^{\mathrm{t}} \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix}$$

$$= 4^n \mathcal{A}^{2n-1} \varphi_2^{-(4n+1)} (\varphi_2^2 + 4a^2 t^2)^{n-1} |x|^2 \sum_{j=1}^{2n} (\mathcal{A} X_j^\ell \varphi_2)^2$$

$$= 4^{n+1} \mathcal{A}^{2n-1} \varphi_2^{-(4n+5)} (\varphi_2^2 + 4a^2 t^2)^n |x|^4.$$

To obtain the final identity above we used the fact that

$$(13) \qquad\qquad\qquad \mathcal{A}^2 \sum_{j=1}^{2n} (X_j^\ell \varphi_2)^2 = 4\varphi_2^{-4} (\varphi_2^2 + 4a^2 t^2) |x|^2.$$

Using fact (11) one more time we see that

$$\mathcal{A}^{4n+2} \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{2n} \\ \mathbf{b}_{2n+1} \end{pmatrix} = 4^n \mathcal{A}^{2n+2} \varphi_2^{-4n} X_{2n+1}^\ell \varphi_2 X_{2n+1}^r \varphi_2 \det(C) + \sum_{j=1}^{2n} \det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{d}_j \\ \vdots \\ \mathbf{c}_{2n} \\ \mathbf{b}_{2n+1} \end{pmatrix}$$

It follows from (12) that for $1 \le j \le n$ we have

$$\det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{d}_j \\ \vdots \\ \mathbf{c}_{2n} \\ \mathbf{b}_{2n+1} \end{pmatrix} = 4^n \mathcal{A}^{2n+1} \varphi_2^{-(4n+1)} X_{2n+1}^\ell \varphi_2 (t X_j^\ell \varphi_2 - 2a\varphi_2 x_{j+n}) \det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{0}_j + 2\varphi_2^{-1} x^{\mathrm{t}} \\ \vdots \\ \mathbf{c}_{2n} \end{pmatrix}$$

$$= 4^n \mathcal{A}^{2n+2} \varphi_2^{-4n} (\varphi_2^2 + 4a^2 t^2)^{n-1} X_{2n+1}^\ell \varphi_2 (t X_j^\ell \varphi_2 - 2a\varphi_2 x_{j+n}) X_j^\ell \varphi_2$$

and similarly

$$\det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{d}_{j+n} \\ \vdots \\ \mathbf{c}_{2n} \\ \mathbf{b}_{2n+1} \end{pmatrix} = 4^n \mathcal{A}^{2n+2} \varphi_2^{-4n} (\varphi_2^2 + 4a^2 t^2)^{n-1} X_{2n+1}^\ell \varphi_2 (t X_{j+n}^\ell \varphi_2 + 2a\varphi_2 x_j) X_{j+n}^\ell \varphi_2.$$

It then follows from the identity

$$\mathcal{A} \sum_{j=1}^{n} \left\{ x_j X_{j+n}^{\ell} \varphi_2 - x_{j+n}) X_j^{\ell} \varphi_2 \right\} = -8a^2 \varphi_2^{-1} t |x|^2$$

together with (13) that

$$\sum_{j=1}^{2n} \det \begin{pmatrix} \mathbf{c}_1 \\ \vdots \\ \mathbf{d}_j \\ \vdots \\ \mathbf{c}_{2n} \\ \mathbf{b}_{2n+1} \end{pmatrix} = 4^{n+1} \mathcal{A}^{2n} \varphi_2^{-(4n+4)} (\varphi_2^2 + 4a^2 t^2)^{n-1} X_{2n+1}^{\ell} \varphi_2 \left\{ (\varphi_2^2 + 4a^2 t^2) - 2\mathcal{A}a^2 \varphi_2^3 \right\} |x|^2 t.$$

Bringing this all together we see that

$$\mathcal{A}^{4n+2} \sum_{j=1}^{2n+1} \det \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{b}_j \\ \vdots \\ \mathbf{a}_{2n+1} \end{pmatrix} = 4^{n+1} \mathcal{A}^{2n} \varphi_2^{-(4n+4)} (\varphi_2^2 + 4a^2 t^2)^n t^2 + \heartsuit$$

where

$$\heartsuit = 4^{n+1} \mathcal{A}^{2n-1} \varphi_2^{-(4n+5)} (\varphi_2^2 + 4a^2 t^2)^{n-1} |x|^2 \left\{ (\varphi_2^2 + 4a^2 t^2)(|x|^2 + 2\varphi_2^{-1} t^2) - 4\mathcal{A}a^2 \varphi_2^2 t^2 \right\}$$

$$= 4^{n+1} \mathcal{A}^{2n} \varphi_2^{-(4n+1)} (\varphi_2^2 + 4a^2 t^2)^{n-1} |x|^2.$$

In the last step we have used the identity

$$\mathcal{A}\varphi_2^2 = |x|^2 + 2\varphi_2^{-1} t^2.$$

We therefore have that

$$\mathcal{A}^{4n+2} \det(\varphi_2 A - \tfrac{\beta+2}{2} B) = -2^{2n+1} \mathcal{A}^{2n-1} \varphi_2^{-(2n+5)} (\varphi_2^2 + 4a^2 t^2)^{n-1} f_2(x,t)$$

where

$$f_2(x,t) = 2\mathcal{A}\varphi_2^4 |x|^2 - (\varphi_2^2 + 4a^2 t^2)(|x|^4 - 2\mathcal{A}\varphi_2 t^2)$$

$$= \mathcal{A}\varphi_2^4 |x|^2 + 4\varphi_2^2 t^2 (1 - a^2) + 16a^2 t^4 + 4a^2 \varphi_2^{-2} t^6.$$

6. Further comparisons

6.1. Nonisotropic $\mathbf{R}^{2n+1}$. When $a = 0$ (not a Heisenberg type group, but still a homogeneous group) we of course have $X_j^{\ell} = X_j^r = \partial/\partial x_j$ and it is then straightforward to verify that in this case we have the following:

If $\rho(x,t) = \rho_1(x,t)$, then (in the notation of Section 5) we get that

$$\det(\varphi_1 A - \tfrac{\beta+4}{4} B) = -b (4\varphi_1)^{2n} |x|^{4n} \left\{ 2(\beta + 1)|x|^4 + 3(\beta + 2)t^2 \right\}$$

and in particular the Hessian degenerates along the line $(0, t)$.

While if $\rho(x,t) = \rho_2(x,t)$, then

$$\mathcal{A}^{4n+2} \det(\varphi_2 A - \tfrac{\beta+2}{2} B) = -2^{2n+1} \mathcal{A}^{2n-1} \varphi_2^{-5} \left\{ \beta \mathcal{A} \varphi_2^3 + \mathcal{A}\varphi_2^2 |x|^2 + 4t^2 \right\}$$

which is clearly non-degenerate. For a closely related result, see Shayya [8].

6.2. **Polarized Heisenberg group.** The polarized Heisenberg group $\mathbf{H}^n_{a,\mathrm{pol}}$ is isomorphic to the full Heisenberg group $\mathbf{H}^n_a$ and has the multiplication law

$$(x,t) \cdot (y,s) = (x+y, s+t - 2a\, x^{\mathrm{t}} J_{\mathrm{pol}}\, y)$$

where again a is a nonzero real number, but now J_{pol} denotes the matrix on $\mathbf{R}^{2n}$,

$$J_{\mathrm{pol}} = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}.$$

In particular, if $n=1$ and $a=-1/2$, then this is the $m=3$ case of the groups of $m \times m$ upper-triangular matrices with ones along the diagonal; see Stein [9].

When $n=1$ the corresponding Lie algebra is generated by the left-invariant vector fields

$$X^\ell_1 = \frac{\partial}{\partial x_1}, \quad X^\ell_2 = \frac{\partial}{\partial x_2} + 2ax_1 \frac{\partial}{\partial t}, \quad \text{and } X^\ell_3 = \frac{\partial}{\partial t},$$

with the right-invariant vector fields being given by

$$X^r_1 = \frac{\partial}{\partial x_1} + 2ax_1 \frac{\partial}{\partial t}, \quad X^r_2 = \frac{\partial}{\partial x_2}, \quad \text{and } X^r_3 = \frac{\partial}{\partial t}.$$

Proposition 10. *Let $n=1$ and $\Phi(x,t) = \rho_1(x,t)^{-\beta}$, then for all real a*

$$\det\left(X^\ell_j X^r_k \Phi(x,t) \right) = 0$$

along the line $(0,t)$.

In actual fact, using again the notation of the previous section, we have that

$$\det(\varphi_1 A - \tfrac{\beta+4}{4}B) = -16\left\{ 2(\beta+1)|x|^8 + 3(\beta+2)|x|^4 t^2 - 2(\beta+2)a\varphi_1 x_1 x_2 t \right\}.$$

We leave the details to the reader.

Proposition 11. *Let $n=1$ and $\Phi(x,t) = \rho_2(x,t)^{-\beta}$ with $\beta > 0$, then for $(x,t) \neq (0,0)$*

$$\det\left(X^\ell_j X^r_k \Phi(x,t) \right) \neq 0$$

provided $a^2 \leq 1$.

Proof. A calculation similar to those above yields

$$\mathcal{A}^6 \det(\varphi_2 A - \tfrac{\beta+2}{2}B) = -8\mathcal{A}\varphi_2^{-7}\left\{ \beta\mathcal{A}\varphi_2(\varphi_2^2 - ax_1 x_2 t) + (2\mathcal{A}\varphi_2^3 - 2\mathcal{A}\varphi_2 ax_1 x_2 t - (x_1^2 + x_2^2)^2) \right\}.$$

Now it is easy to see that

$$\begin{aligned}
2\mathcal{A}\varphi_2^3 - 2\mathcal{A}\varphi_2 ax_1 x_2 t - (x_1^2 + x_2^2)^2 &= 2\varphi_2^2 + 2t^2 - (\varphi_2 - \varphi_2^{-1} t^2)^2 - 2\mathcal{A}\varphi_2 ax_1 x_2 t \\
&\geq \varphi_2^2 - \varphi_2^{-2} t^4 - 2\mathcal{A}\varphi_2 ax_1 x_2 t \\
&= \mathcal{A}\varphi_2(\varphi_2(x_1^2 + x_2^2) - 2ax_1 x_2 t) \\
&\geq \mathcal{A}\varphi_2(x_1^2 + x_2^2)(\varphi_2 - |at|),
\end{aligned}$$

which is clearly nonnegative if $a^2 \leq 1$, since it follows from (8) that $\varphi_2 \geq |t|$. In the last line above we used the easily verifiable inequality

(14) $$2ax_1 x_2 t \leq (x_1^2 + x_2^2)|at|.$$

From this inequality it also follows that

$$\varphi_2^2 - 2ax_1x_2t \geq 2|x_1x_2|(\varphi_2 - |at|)$$

and hence, if we again assume that $a^2 \leq 1$, we see that

$$\varphi_2^2 - ax_1x_2t \geq \varphi_2^2/2. \qquad \qquad \square$$

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