# THE HEISENBERG GROUP FOURIER TRANSFORM

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## 1. Fourier transform on $\mathbf{R}^n$

We start by presenting some standard properties of the Euclidean Fourier transform; see for example [6] and [4].

Given $f \in L^1(\mathbf{R}^n)$, we define its Fourier transform by setting

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

If for $h \in \mathbf{R}^n$ we let $(\tau_h f)(x) = f(x+h)$, then it follows that $\widehat{\tau_h f}(\xi) = e^{ih\cdot\xi}\widehat{f}(\xi)$. Now for suitable $f$ the inversion formula

$$f(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{ix\cdot\xi}\widehat{f}(\xi)d\xi,$$

holds and we see that the Fourier transform decomposes a function into a continuous sum of characters (eigenfunctions for translations).

If $A$ is an orthogonal matrix and $\xi$ is a column vector then $\widehat{f\circ A}(\xi) = \widehat{f}(A\xi)$ and from this it follows that the Fourier transform of a radial function is again radial. In particular the Fourier transform of Gaussians take a particularly nice form; if $G(x) = e^{-|x|^2/2}$, then $\widehat{G}(\xi) = (2\pi)^{\frac{n}{2}}G(\xi)$. In general the Fourier transform of a radial function can always be explicitly expressed in terms of a Bessel transform; if $g(x) = g_0(|x|)$ for some function $g_0$, then

$$\widehat{g}(\xi) = (2\pi)^{\frac{n}{2}} \int_0^\infty g_0(r)(r|\xi|)^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(r|\xi|) r^{n-1} dr,$$

where $J_{\frac{n-2}{2}}$ is a Bessel function.

If $f \in L^2(\mathbf{R}^n)$ then Plancherel's theorem states that $\widehat{f} \in L^2(\mathbf{R}^n)$, more precisely

$$\|\widehat{f}\|_2^2 = (2\pi)^n \|f\|_2^2.$$

If for $f, g \in L^1(\mathbf{R}^n)$ we define convolution by

$$(f * g)(x) = \int f(x-y) g(y) dy,$$

then we have

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

## 2. Fourier analysis on the Heisenberg group

We now turn our attention to the Heisenberg group $\mathbf{H}^n$, see also Stein [5] and the books by Thangavelu [7] and [8].

### 2.1. **Representations of the Heisenberg group.**

The Heisenberg group $\mathbf{H}^n$ is of course $\mathbf{C}^n \times \mathbf{R}$ endowed with the group law

$$[z,t] \cdot [w,s] = [z+w, t+s+\langle z,w \rangle],$$

where the symplectic form $\langle \cdot, \cdot \rangle$ is defined by $\langle z, w \rangle = \frac{1}{2} \operatorname{Im}(z \cdot \bar{w})$, with identity the origin and inverses given by $[z,t]^{-1} = [-z,-t]$.

The following transformations are automorphisms of the group $\mathbf{H}^n$:

- the *nonisotropic dilations* $[z,t] \mapsto \delta \circ [z,t] = [\delta z, \delta^2 t]$, for all $\delta > 0$;
- the *rotations* $[z,t] \mapsto [Uz, t]$, with $U$ a unitary transformation of $\mathbf{C}^n$.

The representation theory of the Heisenberg group is well understood, using the Stone-von Neumann theorem we can give a complete classification of all the irreducible unitary representations of $\mathbf{H}^n$; see Folland [1]. Let $\mathcal{U}(L^2(\mathbf{R}^n))$ denote the group of unitary operators acting on $L^2(\mathbf{R}^n)$, and for each $\lambda \in \mathbf{R}$ define the mapping

$$\pi_\lambda : \mathbf{H}^n \to \mathcal{U}(L^2(\mathbf{R}^n))$$

as follows. For each $(z,t) \in \mathbf{H}^n$, $z = x + iy$, and $\varphi \in L^2(\mathbf{R}^n)$, we let

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y + t)} \varphi(\xi + y).$$

It is then easy to check that $\pi_\lambda(z,t)$ is a homomorphism from $\mathbf{H}^n$ to $\mathcal{U}(L^2(\mathbf{R}^n))$, that is

$$\pi_\lambda(z,t)\pi_\lambda(w,s) = \pi_\lambda(z+w, t+s+\langle z,w \rangle),$$

and that $\pi_\lambda(z,t)$ is unitary. Moreover, it is continuous in the sense that for $\varphi \in L^2$,

$$\|\pi_\lambda(z,t)\varphi - \varphi\|_{L^2} \to 0 \text{ as } (z,t) \to 0.$$

Thus $\pi_\lambda$ is a unitary representation of $\mathbf{H}^n$. Let us now show that they are irreducible when $\lambda \neq 0$. Write $\pi_\lambda(z,t) = \pi_\lambda(z)e^{i\lambda t}$, where $\pi_\lambda(z) = \pi_\lambda(z,0)$. So

$$\pi_\lambda(z)\varphi(\xi) = e^{i\lambda(x\xi+\frac{1}{2}xy)}\varphi(\xi+y).$$

Suppose $M \subset L^2(\mathbf{R})$ is invariant under all $\pi_\lambda(z,t)$. If $M \neq \{0\}$ we will show $M = L^2(\mathbf{R})$ proving the irreducibility of $\pi_\lambda$. Suppose $M$ is a proper invariant subspace of $L^2(\mathbf{R})$, then there exists functions $f, g \in L^2(\mathbf{R}^n)$ such that $f \in M$ and $(\pi_\lambda(z)f, g) = 0$ for all $z$. Now

$$(\pi_\lambda(z)f, g) = \int_{\mathbf{R}^n} e^{i\lambda x\cdot\xi} f(\xi + \frac{y}{2})\bar{g}(\xi - \frac{y}{2})d\xi.$$

Applying the Plancherel theorem for the Fourier transform in the $x$ variable gives

$$\int_{\mathbf{C}^n} |(\pi_\lambda(z)f, g)|^2 dz = \left(\frac{2\pi}{|\lambda|}\right)^n \int_{R^{2n}} |f(\xi + \frac{y}{2})|^2 |\bar{g}(\xi - \frac{y}{2})|^2 d\xi dy$$

which after making a change of variables gives

$$\|(\pi_\lambda(z)f, g)\|^2_{L^2(\mathbf{C}^n)} = \left(\frac{2\pi}{|\lambda|}\right)^n \|f\|^2_{L^2(\mathbf{R}^n)}\|g\|^2_{L^2(\mathbf{R}^n)}, \tag{1}$$

so we must have $\|f\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(\mathbf{R}^2)} = 0$, but this is a contradiction since $f, g$ are non-trivial.

The theorem of Stone-von Neumann, states that, up to unitary equivalence these are all the infinite-dimensional irreducible unitary representations of the Heisenberg group.

It follows from (1) by polarization that if $\varphi, \psi, f, g \in L^2(\mathbf{R}^n)$, then

$$((\pi_\lambda(z)\varphi, \psi), (\pi_\lambda(z)f, g)) = \left(\frac{2\pi}{|\lambda|}\right)^n (\varphi, f)(g, \psi). \tag{2}$$

Notice that this result has the following immediate consequence. Suppose $\{e_j\}$ is an orthonormal system in $L^2(\mathbf{R}^n)$, then $\{e_{jk}\}$ defined by

$$e_{jk} = \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} (\pi_\lambda(z)e_j, e_k)$$

forms an orthonormal system in $L^2(\mathbf{C}^n)$. In actual fact, using properties of the group Fourier transform, we will show that $\{e_{jk}\}$ is an orthonormal basis whenever $\{e_j\}$ is.

2.2. **Group Fourier transform.** Given $f \in L^1(\mathbf{H}^n)$ the group Fourier transform $\widehat{f}(\lambda)$ is defined for each $\lambda \neq 0$ as the operator-valued function on the Hilbert space $L^2(\mathbf{R}^n)$ given by

$$\widehat{f}(\lambda)\varphi(\xi) = \int_{\mathbf{H}^n} f(z,t)\pi_\lambda(z,t)\varphi(\xi)dzdt.$$

If $\psi$ is another function in $L^2(\mathbf{R}^n)$, then

$$(\widehat{f}(\lambda)\varphi, \psi) = \int_{\mathbf{H}^n} f(z,t)(\pi_\lambda(z,t)\varphi, \psi)dzdt.$$

Since $\pi_\lambda(z,t)$ are unitary operators, we have

$$|(\pi_\lambda(z,t)\varphi, \psi)| \leq \|\varphi\|_{L^2(\mathbf{R}^n)}\|\psi\|_{L^2(\mathbf{R}^n)},$$

and so

$$|(\widehat{f}(\lambda)\varphi, \psi)| \leq \|\varphi\|_{L^2(\mathbf{R}^n)}\|\psi\|_{L^2(\mathbf{R}^n)}\|f\|_{L^1(\mathbf{H}^n)}.$$

Hence $\widehat{f}(\lambda)$ is a bounded operator on $L^2(\mathbf{R}^n)$, moreover $\|\widehat{f}(\lambda)\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \leq \|f\|_{L^1(\mathbf{H}^n)}$.

If we define

$$f^\lambda(z) = \int_{\mathbf{R}} e^{i\lambda t} f(z,t)dt,$$

then it is clear that

$$\widehat{f}(\lambda)\varphi(\xi) = W_\lambda(f^\lambda)\varphi(\xi),$$

where[1]

$$W_\lambda(f^\lambda)\varphi(\xi) = \int_{\mathbf{C}^n} f^\lambda(z)\pi_\lambda(z)\varphi(\xi)dz.$$

Letting $z = x + iy = (x, y)$ we see that

$$\begin{aligned}\widehat{f}(\lambda)\varphi(\xi) &= \int_{\mathbf{H}^n} f^\lambda(x,y)e^{i\lambda(x\cdot\xi+\frac{1}{2}x\cdot y)}\varphi(\xi+y)dxdy \\ &= \int_{\mathbf{R}^n} K_\lambda(\xi,\eta)\varphi(\eta)d\eta,\end{aligned}$$

where

$$K_\lambda(\xi,\eta) = \int f^\lambda(x,\eta-\xi)e^{i\lambda(\frac{x}{2}\cdot(\xi+\eta))}dx.$$

We can now prove the Plancherel theorem for the group Fourier transform.

**Theorem 1.** *If $f \in L^2(\mathbf{H}^n)$, then $\widehat{f}(\lambda)$ is a Hilbert-Schmidt operator and*

$$\int \|\widehat{f}(\lambda)\|_{HS}^2|\lambda|^n d\lambda = (2\pi)^{n+1}\int_{\mathbf{H}^n} |f(z,t)|^2 dzdt.$$

*Proof.* Suppose that $f \in L^1 \cap L^2(\mathbf{H}^n)$, it then follows from Plancherel's theorem for the Fourier transform that

$$\|\widehat{f}(\lambda)\|_{HS}^2 = \int_{\mathbf{R}^{2n}} |K_\lambda(\xi,\eta)|^2 d\xi d\eta = \left(\frac{2\pi}{|\lambda|}\right)^n \int_{\mathbf{R}^{2n}} |f^\lambda(x,y)|^2 dxdy.$$

If we now integrate in $\lambda$ and using Plancherel's theorem for the Fourier transform in the $t$ variable we get

$$\begin{aligned}\int \|\widehat{f}(\lambda)\|_{HS}^2|\lambda|^n d\lambda &= \int (2\pi)^n \int_{\mathbf{C}^n} \left|\int_{\mathbf{R}} e^{i\lambda t}f(z,t)dt\right|^2 dzd\lambda \\ &= (2\pi)^n \int_{\mathbf{C}^n}\int \left|\int_{\mathbf{R}} e^{i\lambda t}f(z,t)dt\right|^2 d\lambda dz, \\ &= (2\pi)^{n+1}\int_{\mathbf{H}^n} |f(z,t)|^2 dzdt.\end{aligned}$$

An additional limiting argument then proves the theorem. □

**Corollary 2.** *If $\{e_j\}$ is an orthonormal basis for $L^2(\mathbf{R}^n)$, then $\{e_{jk}\}$ defined by*

$$e_{jk} = \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} (\pi_\lambda(z)e_j, e_k),$$

*is an orthonormal basis for $L^2(\mathbf{C}^n)$.*

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[1] When $\lambda = 1$ this is called the Weyl transform of $f$.

*Proof.* Orthonormality follows immediately from (2), to prove completeness suppose $g \in L^2(\mathbf{C}^n)$ is orthogonal to all $e_{jk}$, that is

$$\int_{\mathbf{C}^n} \bar{g}(z)(\pi_\lambda(z)e_j, e_k)dz = 0.$$

If we now take $h \in L^2(\mathbf{R})$ and define $f(z,t) = \bar{g}(z)h(t)$, it follows that

$$(\widehat{f}(\lambda)e_j, e_k) = \int_{\mathbf{H}^n} f(z,t)(\pi_\lambda(z,t)e_j, e_k)dzdt = \int_{\mathbf{R}} h(t)e^{i\lambda t}dt \int_{\mathbf{C}^n} \bar{g}(z)(\pi_\lambda(z)e_j, e_k)dz = 0.$$

The completeness of $\{e_k\}$ in $L^2(\mathbf{R}^n)$ then implies $\widehat{f}(\lambda)e_j = 0$ which shows $\widehat{f}(\lambda) = 0$. Plancherel's theorem for the group Fourier transform then implies that $f = 0$. □

2.3. **Convolution and twisted convolution.** Consider the operator $f \mapsto f*K$ where convolution is taken with respect to the group structure on $\mathbf{H}^n$, that is

$$\int_{\mathbf{H}^n} f(w,s)K([w,s]^{-1} \cdot [z,t])dwds.$$

Using the fact that $\pi_\lambda(z,t)$ is a homomorphism from $\mathbf{H}^n$ to the group of unitary operators on $L^2(\mathbf{R}^n)$, we see that

$$\widehat{f * K}(\lambda) = \widehat{f}(\lambda)\widehat{K}(\lambda).$$

Now it follows from Plancherel's theorem for the group Fourier transform that the boundedness of our convolution operators on $L^2(\mathbf{H}^n)$ is equivalent to the uniform boundedness of the operator norms of $\widehat{K}(\lambda)$ over $\lambda \neq 0$. Note that in the previous section we saw

$$\widehat{K}(\lambda)\varphi(\xi) = \int_{\mathbf{R}^n} L_\lambda(\xi,\eta)\varphi(\eta)d\eta,$$

where

$$L_\lambda(\xi,\eta) = \iint K(x, \eta-\xi, t)e^{i\lambda(\frac{x}{2}\cdot(\xi+\eta)+t)}dxdt.$$

Alternatively, in view of the Plancherel theorem for the Fourier transform, the boundedness of our convolution operators on $L^2(\mathbf{H}^n)$ is also equivalent to the estimate

$$\int_{\mathbf{C}^n} |(f*K)^\lambda(z)|^2 dz \leq C \int_{\mathbf{C}^n} |f^\lambda(z)|^2 dz,$$

where the constant $C$ is independent of $\lambda$. Now as

$$(f*K)^\lambda(z) = \int_{\mathbf{C}^n} f^\lambda(w)K^\lambda(z-w)e^{i\lambda\langle z,w\rangle}dw,$$

we are naturally led to consider so called *twisted* convolutions

$$(g *_\lambda h)(z) = \int_{\mathbf{C}^n} g(w)h(z-w)e^{i\lambda\langle z,w\rangle}dw.$$

It then follows that boundedness of our convolution operators on $L^2(\mathbf{H}^n)$ is equivalent to that of the twisted convolution operator $f \mapsto f *_\lambda K^\lambda$ on $L^2(\mathbf{C}^n)$. Note that

$$(f *_\lambda K^\lambda)(z) = \int_{\mathbf{C}^n} M_\lambda(z,w)f(w)dw,$$

where

$$M_\lambda(z,w) = \int K(z-w,t)e^{i\lambda(t+\langle z,w\rangle)}dt.$$

Note also that using the fact that

$$\pi_\lambda(z)\pi_\lambda(w) = \pi_\lambda(z+w)e^{i\lambda\langle z,w\rangle},$$

it is easy to see that the operator $W_\lambda$ bears the same relation to the twisted convolution as the group Fourier transform bears to the convolution, namely

$$W_\lambda(g *_\lambda h) = W_\lambda(g)W_\lambda(h).$$

However, if our kernels were chosen radial on $\mathbf{H}^n$, i.e. $K(z,t) = K_0(|z|,t)$ for some function $K_0$, then it is a result of Geller [3] that the operators $\widehat{K}(\lambda)$ are in fact diagonal on the Hermite basis for $L^2(\mathbf{R}^n)$ and that the diagonal entries can be expressed explicitly in terms a Laguerre functions. Therefore, for radial $K$, studying the boundedness of convolution operators on $L^2(\mathbf{H}^n)$ reduces to studying the uniform behavior of these diagonal entries.

The remainder of this appendix shall be devoted to the formulation and proof of this result, we start by introducing the Hermite and Laguerre functions.

## 3. Hermite and Laguerre functions

Fourier analysis on the Heisenberg group is intimately connected with Hermite expansions in $\mathbf{R}^n$.

3.1. **Hermite polynomials.** Hermite polynomials are defined, for $x \in \mathbf{R}$, by

$$H_k(x) = (-1)^k e^{x^2}\frac{d^k}{dx^k}e^{-x^2}.$$

By expanding $e^{-(x-r)^2}$ in a Taylor series, multiply by $e^{x^2}$ and then compare terms one obtains the following generating function identity. If $|r|<1$, then

$$\sum_{k=0}^\infty H_k(x)\frac{r^k}{k!} = e^{x^2}e^{-(x-r)^2}.$$

**Proposition 3.** *The Hermite polynomials $\{H_k\}_{k=0}^\infty$ form a complete orthogonal set on $\mathbf{R}$ with respect to the weight $w(x) = e^{-x^2}$ and $\|H_k\|_w^2 = 2^k k!\pi^{\frac{1}{2}}$.*

*Proof.* If $P$ is any polynomial, then

$$\int_{\mathbf{R}} P(x)H_k(x)e^{-x^2}dx = (-1)^k\int_{\mathbf{R}} P(x)\frac{d^k}{dx^k}e^{-x^2}dx = \int_{\mathbf{R}} P^{(k)}(x)e^{-x^2}dx.$$

Now if $P$ is a polynomial of degree less that $k$, in particular if $P = H_j$ with $j<k$ then $P^{(k)} \equiv 0$ and the orthogonality of the Hermite polynomials follows. Now if $P = H_k$, then we have $P(x) = (2x)^k + \dots$ and hence $P^{(k)} \equiv 2^k k!$ and

$$\|H_k\|_w^2 = 2^k k!\int_{\mathbf{R}} e^{-x^2}dx = 2^k k!\pi^{\frac{1}{2}}.$$

Finally we point out that our family $\{H_k\}$ is complete in $L^2(\mathbf{R})$. That is, if $f \in L^2(\mathbf{R})$ and

$$\int_{\mathbf{R}} f(x) H_k(x) e^{-x^2} dx = 0, \text{ for all } k,$$

then $f = 0$. This is equivalent to

$$I(r) = \int_{\mathbf{R}} f(x) e^{-x^2} e^{2rx} dx = 0.$$

Now, $I(r)$ is entire, so this equality holds for purely imaginary $r$. The inverse Fourier transform applied to $f(x)e^{-x^2}$ then gives that $f = 0$. $\square$

It is therefore clear that $h_k(x) = H_k(x) e^{-\frac{x^2}{2}}$ forms an orthogonal basis for $L^2(\mathbf{R})$ and furthermore that for each $\lambda \neq 0$ the normalized, rescaled Hermite functions

$$h_k^\lambda(x) = (2^k k!)^{-\frac{1}{2}} \left(\frac{|\lambda|}{\pi}\right)^{\frac{1}{4}} h_k(|\lambda|^{\frac{1}{2}} x),$$

form an orthonormal basis. It follows that if we now have $x \in \mathbf{R}^n$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ then the functions

$$h_\alpha^\lambda(x) = h_{\alpha_1}^\lambda(x_1) \ldots h_{\alpha_n}^\lambda(x_n),$$

forms an orthonormal basis for $L^2(\mathbf{R}^n)$, where $\alpha$ ranges over all multi–indices.

Now given any function $f$ on $\mathbf{R}^n$, the Hermite expansion of $f$ is (formally) given by

$$f(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} (f, h_\alpha^\lambda) h_\alpha^\lambda(x) = \sum_{k=0}^{\infty} P_k f(x),$$

where $P_k$ is the projection onto the $k$th eigenspace spanned by $\{h_\alpha^\lambda : |\alpha| = k\}$. Of course $P_k$ is an integral operator with kernel

$$h_k^\lambda(x, y) = \sum_{|\alpha|=k} h_\alpha^\lambda(x) h_\alpha^\lambda(y).$$

**Lemma 4.** *For $|r| < 1$ we have the following identity known as Mehler's formula, for $x, y \in \mathbf{R}$*

$$\sum_{k=0}^{\infty} r^k h_k^\lambda(x) h_k^\lambda(y) = \left(\frac{|\lambda|}{\pi}\right)^{1/2} (1 - r^2)^{-1/2} e^{\frac{|\lambda|}{2}(x^2 - y^2)} e^{-|\lambda| \frac{(x-ry)^2}{1-r^2}}.$$

*Proof.* It clearly suffices to prove the following generating function identity for $H_k$; for $|r| < 1$,

$$\sum_{k=0}^{\infty} H_k(x) H_k(y) \frac{r^k}{2^k k!} = (1 - r^2)^{-1/2} e^{x^2} e^{-\frac{(x-ry)^2}{1-r^2}}.$$

To see this first recall

$$\int_{\mathbf{R}} e^{-i2ux} e^{-u^2} du = \pi^{1/2} e^{-x^2}.$$

Now,

$$\begin{aligned}
H_k(x) &= (-1)^k e^{x^2}\left(\frac{d}{dx}\right)^k e^{-x^2} \\
&= (-1)^k e^{x^2}\pi^{-1/2}\left(\frac{d}{dx}\right)^k \int_{\mathbf{R}} e^{-i2ux}e^{-u^2}du \\
&= e^{x^2}\pi^{-1/2}\int_{\mathbf{R}}(i2u)^k e^{-i2ux}e^{-u^2}du.
\end{aligned}$$

So,

$$\begin{aligned}
\sum_{k=0}^{\infty} H_k(x)H_k(y)\frac{r^k}{2^k k!} &= \frac{1}{\pi}e^{x^2+y^2}\iint \sum_{k=0}^{\infty}\frac{(-2uvr)^k}{k!}e^{-i2ux}e^{-u^2}e^{-i2vy}e^{-v^2}dudv \\
&= \frac{1}{\pi}e^{x^2+y^2}\int e^{-i2ux}e^{-u^2}\left(\int e^{-i2vy}e^{-2uvr}e^{-v^2}dv\right)du.
\end{aligned}$$

But,

$$\int e^{-i2vy}e^{-2uvr}e^{-v^2}dv = e^{u^2r^2}e^{i2ury}\int_{\mathbf{R}} e^{-i2vy}e^{-v^2}dv = \pi^{1/2}e^{u^2r^2}e^{i2ury}e^{-y^2}.$$

Therefore,

$$\begin{aligned}
\sum_{k=0}^{\infty} H_k(x)H_k(y)\frac{r^k}{2^k k!} &= \pi^{-1/2}e^{x^2}\int e^{-i2ux}e^{-u^2}e^{u^2r^2}e^{i2ury}du \\
&= \pi^{-1/2}e^{x^2}\int e^{-u^2(1-r^2)}e^{-i2u(x-ry)}du \\
&= \pi^{-1/2}e^{x^2}(1-r^2)^{-1/2}\int e^{-u^2}e^{-i2u\frac{(x-ry)}{(1-r^2)^{1/2}}}du \\
&= (1-r^2)^{-1/2}e^{x^2}e^{-\frac{(x-ry)^2}{1-r^2}}.
\end{aligned}$$

□

The Mehler kernel is defined, for $|r| < 1$, by

$$M(x,y,r) = \sum_{k=0}^{\infty} r^k h_k^{\lambda}(x,y).$$

It therefore follows from Mehler's formula that,

$$M(x,y,r) = \left(\frac{|\lambda|}{\pi}\right)^{n/2}(1-r^2)^{-n/2}e^{-\frac{|\lambda|}{2(1-r^2)}((|x|^2+|y|^2)(1+r^2)-4rx\cdot y)}.$$

*Remark* 5. The Hermite functions $h_\alpha^\lambda$ are the normalized eigenfunctions of the operator $\lambda^2|x|^2-\Delta$, corresponding to the eigenvalues $|\lambda|(2|\alpha|+n)$.

3.2. **Laguerre polynomials.** Laguerre polynomials of type $\delta > -1$ are defined, for $x \in (0, \infty)$, by

$$L_k^\delta(x) = e^x x^{-\delta} \frac{1}{k!} \frac{d^k}{dx^k}(e^{-x} x^{k+\delta}).$$

These are clearly polynomials of degree $k$ and from the product rule for derivatives it follows that

$$L_k^\delta(x) = \sum_{j=0}^{k} \binom{k+\delta}{k-j} \frac{(-x)^j}{j!}.$$

From this it is easy to see that

$$\frac{d}{dx} L_k^\delta(x) = -L_{k-1}^{\delta+1}(x).$$

We also have the following generating function for the Laguerre polynomials; for $x > 0$ and $|r| < 1$,

$$\sum_{k=0}^{\infty} L_k^\delta(x) r^k = (1-r)^{-\delta-1} e^{-\frac{r}{1-r}x}. \tag{3}$$

**Proposition 6.** *The Laguerre polynomials* $\{L_k^\delta\}_{k=0}^\infty$ *form a complete orthogonal set on* $(0, \infty)$ *with respect to the weight* $w(x) = e^{-x} x^\delta$ *and* $\|L_k^\delta\|_w^2 = \frac{(k+\delta)!}{k!}$.

*Proof.* If $P$ is any polynomial, then

$$\int_0^\infty P(x) L_k^\delta(x) e^{-x} x^\delta dx = \frac{1}{k!} \int_0^\infty P(x) \frac{d^k}{dx^k} e^{-x} x^{k+\delta} dx = \frac{(-1)^k}{k!} \int_0^\infty P^{(k)}(x) e^{-x} x^{k+\delta} dx.$$

Now if $P$ is a polynomial of degree less that $k$, in particular if $P = L_j^\delta$ with $j < k$ then $P^{(k)} \equiv 0$; this proves the orthogonality of the Laguerre polynomials. Now if $P = L_k^\delta$, then we have $P^{(k)} = (-1)^k$ and

$$\|L_k^\delta\|_w^2 = \frac{1}{k!} \int_0^\infty e^{-x} x^{k+\delta} dx = \frac{(k+\delta)!}{k!}.$$

The argument giving completeness is similar to that for Hermite polynomials; see Folland [2]. $\square$

**Definition.** Laguerre functions of type $\delta$, $\delta > -1$ are given by

$$\Lambda_k^\delta(x) = \left(\tfrac{k!}{(k+\delta)!}\right)^{1/2} L_k^\delta(x) e^{-\frac{1}{2}x} x^{\frac{\delta}{2}}.$$

It is an immediate consequence of Proposition 6 that $\Lambda_k^\delta(x)$ form an orthonormal basis for $L^2(\mathbf{R}^+)$. In particular, we also have that for each $\lambda \neq 0$, the "Laguerre functions"

$$\ell_k^\lambda(r) = (|\lambda| r^2)^{\frac{1-n}{2}} \Lambda_k^{n-1}(\tfrac{1}{2}|\lambda| r^2),$$

form an orthonormal basis for $L^2(\mathbf{R}^+, |\lambda|^n r^{2n-1} dr)$.

3.3. **Special Hermite functions.** For each $\alpha, \beta \in \mathbf{N}^n$ and $z \in \mathbf{C}^n$, we define the special Hermite functions $h_{\alpha,\beta}^\lambda$ by

$$h_{\alpha,\beta}^\lambda(z) = \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} (\pi_\lambda(z) e_j, e_k).$$

We have therefore already shown that $\{h_{\alpha,\beta}^\lambda\}$ forms an orthonormal basis in $L^2(\mathbf{C}^n)$.

For a function $f \in L^2(\mathbf{C}^n)$ we have the eigenfunction expansion

$$f = \sum_\alpha \sum_\beta (f, h^\lambda_{\alpha,\beta}) h^\lambda_{\alpha,\beta}$$

which is called the special Hermite expansion. This series can be put in a compact form once we've shown that special Hermite functions can be expressed in terms of Laguerre functions.

**Proposition 7.** *For $f \in L^2(\mathbf{C}^n)$ we have*

$$f(z) = \sum_{k=0}^\infty f *_\lambda \varphi^\lambda_k(z),$$

*where* $\varphi^\lambda_k(z) = (\frac{|\lambda|}{2\pi})^n L^{n-1}_k(\frac{1}{2}|\lambda||z|^2)e^{-\frac{1}{4}|\lambda||z|^2}$.

This will be an immediate consequence of the following two lemmas. The first will show that our Laguerre functions are in fact special Hermite functions, more precisely

**Lemma 8.**

$$h^\lambda_{\alpha,\alpha}(z) = \left(\frac{|\lambda|}{2\pi}\right)^{n/2} \prod_{j=1}^n L^0_{\alpha_j}(\tfrac{1}{2}|\lambda||z_j|^2)e^{-\frac{1}{4}|\lambda||z_j|^2}.$$

*Proof.* It suffices to consider the one dimensional case. Mehler's kernel identity gives,

$$\sum_{k=0}^\infty h^\lambda_k(x)h^\lambda_k(y)r^k = \left(\frac{|\lambda|}{\pi}\right)^{1/2}(1-r^2)^{-1/2}e^{-\frac{|\lambda|}{2(1-r^2)}((x^2+y^2)(1+r^2)-4xyr)}.$$

An easy calculation gives that

$$\sum_{k=0}^\infty h^\lambda_k(\xi+\frac{y}{2})h^\lambda_k(\xi-\frac{y}{2})r^k = \left(\frac{|\lambda|}{\pi}\right)^{1/2}(1-r^2)^{-1/2}e^{-|\lambda|\left(\xi^2\frac{1-r}{1+r}+\frac{y^2}{4}\frac{1+r}{1-r}\right)}.$$

Therefore,

$$\begin{aligned}
\sum_{k=0}^\infty h^\lambda_{kk}(z)r^k &= \left(\frac{|\lambda|}{2\pi}\right)^{1/2}\sum_{k=0}^\infty(\pi_\lambda(z)h^\lambda_k, h^\lambda_k))r^k \\
&= \left(\frac{|\lambda|}{2\pi}\right)^{1/2}\int_{\mathbf{R}} e^{i\lambda x\xi}\sum_{k=0}^\infty h^\lambda_k(\xi+\frac{y}{2})h^\lambda_k(\xi-\frac{y}{2})r^k d\xi \\
&= \frac{|\lambda|}{\pi}(2(1-r^2))^{-1/2}e^{-|\lambda|\frac{y^2}{4}\frac{1+r}{1-r}}\int_{\mathbf{R}} e^{-|\lambda|\xi^2\frac{1-r}{1+r}}e^{i|\lambda|x\xi}d\xi \\
&= \left(\frac{|\lambda|}{2\pi}\right)^{1/2}(1-r)^{-1}e^{-\frac{|\lambda|}{4}\frac{1+r}{1-r}(x^2+y^2)}.
\end{aligned}$$

But it follows from (3) that

$$\sum_{k=0}^\infty L^0_k(x)e^{-\frac{1}{2}x}r^k = (1-r)^{-1}e^{-\frac{1+r}{2(1-r)}x}.$$

Hence,

$$\sum_{k=0}^{\infty} h_{kk}^{\lambda}(z)r^k = \left(\frac{|\lambda|}{2\pi}\right)^{1/2} \sum_{k=0}^{\infty} L_k^0(\frac{1}{2}|\lambda|(x^2+y^2))e^{-\frac{1}{4}|\lambda|(x^2+y^2)}r^k.$$

□

**Lemma 9.**

$$h_{\alpha,\beta}^{\lambda} *_{\lambda} h_{\mu,\nu}^{\lambda} = \left(\frac{2\pi}{|\lambda|}\right)^{n/2} \delta_{\beta,\mu} h_{\alpha,\nu}^{\lambda},$$

*Proof.* For $\varphi, \psi \in L^2(\mathbf{R}^n)$,

$$\begin{aligned}(W_\lambda(\bar{h}_{\alpha,\beta}^{\lambda})\varphi,\psi) &= \int_{\mathbf{C}^n} \bar{h}_{\alpha,\beta}^{\lambda}(\pi_\lambda(z)\varphi,\psi)dz \\ &= \left(\frac{|\lambda|}{2\pi}\right)^{n/2}((\pi_\lambda(z)\varphi,\psi),(\pi_\lambda(z)h_\alpha^\lambda,h_\beta^\lambda)) \\ &= \left(\frac{2\pi}{|\lambda|}\right)^{n/2}(\varphi,h_\alpha^\lambda)(h_\beta^\lambda,\psi).\end{aligned}$$

Therefore

$$(W_\lambda(\bar{h}_{\alpha,\beta}^{\lambda} *_\lambda \bar{h}_{\mu,\nu}^{\lambda})\varphi,\psi) = \left(\frac{2\pi}{|\lambda|}\right)^{n/2}(W_\lambda(\bar{h}_{\mu,\nu}^{\lambda})\varphi,h_\alpha^\lambda)(h_\beta^\lambda,\psi) = \left(\frac{2\pi}{|\lambda|}\right)^{n}(\varphi,h_\mu^\lambda)(h_\nu^\lambda,h_\alpha^\lambda)(h_\beta^\lambda,\psi).$$

This of course implies

$$\bar{h}_{\alpha,\beta}^{\lambda} *_\lambda \bar{h}_{\mu,\nu}^{\lambda} = \left(\frac{2\pi}{|\lambda|}\right)^{n/2} \delta_{\alpha,\nu}\bar{h}_{\mu,\beta}^{\lambda},$$

which gives us our result, since $\overline{f *_\lambda g} = \bar{g} *_\lambda \bar{f}$. □

*Proof of Proposition 7.* For $f \in L^2(\mathbf{C}^n)$ we have that

$$f = \sum_\alpha \sum_\beta (f, h_{\alpha,\beta}^\lambda)h_{\alpha,\beta}^\lambda.$$

Lemma 9 therefore implies that

$$f *_\lambda h_{\mu,\mu}^\lambda = \sum_\alpha \sum_\beta (f,h_{\alpha,\beta}^\lambda)h_{\alpha,\beta}^\lambda *_\lambda h_{\mu,\mu}^\lambda = \left(\frac{2\pi}{|\lambda|}\right)^{n/2} \sum_\alpha (f,h_{\alpha,\mu}^\lambda)h_{\alpha,\mu}^\lambda, \tag{4}$$

and so

$$f(z) = \left(\frac{|\lambda|}{2\pi}\right)^{n/2} \sum_{k=0}^{\infty} \sum_{|\beta|=k} f *_\lambda h_{\beta,\beta}^\lambda(z).$$

We are therefore required to show that

$$\sum_{|\beta|=k} h_{\beta,\beta}^\lambda(z) = \left(\frac{2\pi}{|\lambda|}\right)^{n/2} \varphi_k^\lambda(z). \tag{5}$$

It follows from (3) that $\varphi_k^\lambda$ satisfy the generating function

$$\sum_{k=0}^{\infty} \varphi_k^\lambda(z) r^k = \left(\frac{|\lambda|}{2\pi}\right)^n (1-r)^{-n} e^{-\frac{1}{4}|\lambda||z|^2 \frac{1+r}{1-r}}. \tag{6}$$

Now it follows from Lemma 8 that

$$h_{\beta,\beta}^\lambda(z) = \left(\frac{|\lambda|}{2\pi}\right)^{n/2} \prod_{j=1}^{n} L_{\beta_j}^0(\tfrac{1}{2}|\lambda||z_j|^2) e^{-\frac{1}{4}|\lambda||z_j|^2},$$

and from (3) we see that each $L_{\beta_j}^0(\frac{1}{2}|\lambda||z_j|^2)e^{-\frac{1}{4}|\lambda||z_j|^2}$ satisfies

$$\sum_{k=0}^{\infty} L_k^0(\tfrac{1}{2}|\lambda||z_j|^2) e^{-\frac{1}{4}|\lambda||z_j|^2} r^k = (1-r)^{-1} e^{-\frac{1}{4}|\lambda||z_j|^2 \frac{1+r}{1-r}}.$$

From this it is clear that

$$\sum_{k=0}^{\infty} \sum_{|\beta|=k} h_{\beta,\beta}^\lambda(z) r^k = \left(\frac{|\lambda|}{2\pi}\right)^{n/2} (1-r)^{-n} e^{-\frac{1}{4}|\lambda||z|^2 \frac{1+r}{1-r}},$$

comparing this with (6) we obtained our desired result. □

**Lemma 10.**

$$W_\lambda(\varphi_k^\lambda) = P_k \qquad \textit{and hence} \qquad \varphi_k^\lambda *_\lambda \varphi_j^\lambda = \delta_{kj}\varphi_k^\lambda.$$

*Proof.* From the calculation in the proof of Lemma 9 it follows that

$$W_\lambda(h_{\alpha,\alpha}^\lambda)\varphi = \left(\frac{2\pi}{|\lambda|}\right)^{n/2} (\varphi, h_\alpha^\lambda) h_\alpha^\lambda,$$

in view of identity (5) we therefore have

$$W_\lambda(\varphi_k^\lambda)\varphi = \sum_{|\alpha|=k} (\varphi, h_\alpha^\lambda) h_\alpha^\lambda = P_k,$$

and the second part of the claim follows immediately. □

## 4. Group Fourier transform of radial functions on the Heisenberg group

Recall that the group Fourier transform of an integrable function $g$ on $\mathbf{H}^n$ is, for each $\lambda \neq 0$, an operator-valued function on the Hilbert space $L^2(\mathbf{R}^n)$ given by

$$\widehat{g}(\lambda)\varphi(\xi) = W_\lambda(g^\lambda)\varphi(\xi),$$

where

$$W_\lambda(f^\lambda)\varphi(\xi) = \int_{\mathbf{C}^n} g^\lambda(z)\pi_\lambda(z)\varphi(\xi)dz \quad \text{and} \quad g^\lambda(z) = \int_{\mathbf{R}} g(z,t)e^{i\lambda t}dt.$$

Now if $g$ is also radial on $\mathbf{H}^n$, which means that it depends only on $|z|$ and $t$, then it follows that the operators $\widehat{g}(\lambda)$ are diagonal on the Hermite basis for $L^2(\mathbf{R}^n)$.

**Theorem 11.** *If $g \in L^1(\mathbf{H}^n)$ and $g(z,t) = g_0(|z|,t)$, then*

$$\widehat{g}(\lambda)h_\alpha^\lambda(x) = C_n \mu(|\alpha|,\lambda)h_\alpha^\lambda(x),$$

*where*

$$\mu(k,\lambda) = \big(\tfrac{k!}{(k+n-1)!}\big)^{1/2} \int_0^\infty g_0^\lambda(s)(\tfrac{1}{2}|\lambda|s^2)^{\frac{1-n}{2}} \Lambda_k^{n-1}(\tfrac{1}{2}|\lambda|s^2)s^{2n-1}ds,$$

*and $C_n$ is a constant which depends only on $n$.*

*Proof.* It is clear that $g^\lambda(z) = g_0^\lambda(|z|)$, for some function $g_0^\lambda$. We can therefore write

$$g_0^\lambda(r) = \sum_{k=0}^\infty \Big(\int_0^\infty g_0^\lambda(s)\ell_k^\lambda(s)|\lambda|^n s^{2n-1}ds\Big)\ell_k^\lambda(r).$$

From this we see that we formally have

$$g^\lambda(z) = C_n \sum_{k=0}^\infty \mu(k,\lambda)\varphi_k^\lambda(z),$$

where $C_n = (2\pi)^n 2^{1-n}$. It now follows from Lemma 10 that

$$g^\lambda *_\lambda \varphi_k^\lambda(z) = C_n\mu(k,\lambda)\varphi_k^\lambda(z),$$

and hence from Proposition 7 we see that this formal "Laguerre expansion" in fact agrees with the special Hermite expansion,

$$g^\lambda(z) = \sum_{k=0}^\infty g^\lambda *_\lambda \varphi_k^\lambda(z) = C_n \sum_{k=0}^\infty \mu(k,\lambda)\varphi_k^\lambda(z). \tag{7}$$

Now since $\widehat{g}(\lambda) = W_\lambda(g^\lambda)$ we see that Theorem 11 follow immediately from (7) and Lemma 10. □

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