Math 8440 - Arithmetic Combinatorics - Spring 2011

RAMSEY THEORY

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1. Introduction

A typical result in (additive) Ramsey theory takes the following form: if $\mathbb{N}$ (or $\{1,\ldots,N\}$ with N sufficiently large) is partitioned into finitely many classes, then at least one of these classes will contain contain a specific arithmetic structure (e.g. an arithmetic progression). The simplest example of such a result is the *pigeonhole principle* and one can view Ramsey theory as the study of generalizations and repeated applications of this principle. These notes follow the excellent presentations in [3], [8], [5], [7] and [6] closely.

1.1. Three classical theorems.

The following results actually pre-date Ramsey's theorem itself.

Theorem 1.1 (Hilbert, 1892)**.** *Let $k \in \mathbb{N}$. If $\mathbb{N}$ is finitely colored, then there exist in one color class infinitely many translates of a set of the following form:*

$$\left\{\sum_{i\in I} x_i \,:\, I \subseteq \{1,\ldots,k\}\right\}$$

with $x_1,\ldots,x_k \in \mathbb{N}$.

Note specifically (the case $k=2$ above) that one color class will contain a set of the form

$$\{a, a+x_1, a+x_2, a+x_1+x_2\}, \quad x_1, x_2 \neq 0.$$

Theorem 1.2 (Schur, 1916 (for the case $k=2$))**.** *Let $k \in \mathbb{N}$. If $\mathbb{N}$ is finitely colored, then there exist in one color class infinitely many $(k+1)$-tuples of the form:*

$$\{x_1,\ldots,x_k, x_1+\cdots+x_k\}.$$

Note specifically (the case $k=2$ above) that one color class will contain a triple of the form

$$\{x_1, x_2, x_1+x_2\}.$$

Theorem 1.3 (van der Waerden, 1927)**.** *Let $k \in \mathbb{N}$. If $\mathbb{N}$ is finitely colored, then there exist in one color class infinitely many arithmetic progressions of length k, namely sets of the form*

$$\{a, a+h, \ldots, a+(k-1)h\}$$

with $h \neq 0$.

Note specifically (the case $k=2$ above) that one color class will contain a set of the form

$$\{a, a+h, a+2h\}, \quad h \neq 0.$$

1.2. Gallai's theorem.

"While in Khintchine's book [4] van der Waerden's theorem is called a *pearl of number theory*, it should, perhaps, be more properly called a pearl of geometry" - Vitaly Bergelson [1].

Indeed, it is not hard to see that van der Waerden's theorem is equivalent to the following more overtly geometric result, which is also suggestive of natural multidimensional extensions.

Theorem 1.4. *Let $F \subset \mathbb{N}$ be finite. If $\mathbb{N}$ is finitely colored, then there exist in one color class infinitely many homothetic copies of F, that is infinitely many sets of the form $a+hF = \{a+hx \,:\, x \in F\}$ with $h \neq 0$.*

In Section 8 we shall present a proof of the Hales-Jewett theorem, an abstract combinatorial generalization of van der Waerden's theorem, from which we will easily be able to deduce the following formulation of a multidimensional analogue of Theorem 1.4, originally due to Gallai.

Theorem 1.5 (Gallai's theorem)**.** *Let $n \in \mathbb{N}$ and $F \subset \mathbb{N}^n$ be finite. If $\mathbb{N}^n$ is finitely colored, then there exist in one color class infinitely many homothetic copies of F, that is infinitely many sets of the form $a+hF = \{a+hx \,:\, x \in F\}$ with $a \in \mathbb{N}^n$ and $h \in \mathbb{N}$.*

1.3. **Rado's theorem.** Schur's theorem generalizes considerably. One may interpret Schur's theorem as a result about the existence in one color class of a solution to the linear equation $x_1 + \cdots + x_k = x_{k+1}$. For any $k \in \mathbb{N}$ and homogeneous linear equation $c_1x_1 + \cdots + c_kx_k = 0$, we may ask whether every finite coloring of $\mathbb{N}$ must admit a monochromatic solution. In 1933, Rado (Schur's student) proved that such a linear equation will in fact have this property if and only if some non-empty subset of $\{c_1, \ldots, c_k\}$ sums to zero. More generally, he answered this question for a system of homogeneous linear equations, determining necessary and sufficient conditions for the existence of monochromatic solutions.

Definition 1.6. A matrix C, whose entries are all rational numbers, is said to satisfy the *columns condition* if there is an ordering $c_1, \ldots, c_k$ of its column vectors and a partition of C into blocks of consecutive columns such that the sum of the columns in any block is a linear combination of the columns in the preceding blocks.

Some examples:

1. The matrix $(1, 1, -1)$ satisfies the columns condition. More generally, the matrix $(c_1, \ldots, c_k)$ satisfies the columns condition if and only if some non-empty subset of $\{c_1, \ldots, c_k\}$ sums to zero.
2. Matrices of the form
$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$
can also easily be seen to satisfy the columns condition.

Theorem 1.7 (Rado, 1933)**.** *Let C be matrix whose entries are all rational numbers. If $\mathbb{N}$ is finitely colored, then there exist in one color class infinitely many solutions to the homogeneous system of linear equations $Cx = 0$ if and only if C satisfies the* columns condition.

Note specifically, in light of the examples above, that Rado's theorem implies both Schur's theorem and van der Waerden's theorem as well as the following strengthening of van der Waerden's theorem: if $\mathbb{N}$ is finitely colored, then one color class will contain an arbitrarily long arithmetic progression and its difference.

In Section 6 we shall prove Rado's theorem for the special case of systems with one equation. For a proof of the complete theorem see [3] or [5].

1.4. **Folkman's theorem.** In Section 7, we will use van der Waerden's theorem to establish the following generalization of Schur's theorem (and Hilbert's theorem). The alert reader will also notice that this result may be derived as a corollary of Rado's theorem (the full version), the direct proof we present is however much simpler.

Theorem 1.8 (Folkman's theorem)**.** *Let $k \in \mathbb{N}$. If $\mathbb{N}$ is finitely colored, then there exist in one color class infinitely many sets of the form*
$$\left\{ \sum_{i \in I} x_i \,:\, I \subseteq \{1, \ldots, k\},\, I \neq \emptyset \right\}$$
with $x_1, \ldots, x_k$ distinct.

Note specifically (the case $k = 3$ above) that one color class will contain a set of the form
$$\{x_1, x_2, x_3, x_1 + x_2, x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3\}$$
with $x_1, \ldots, x_3$ distinct and that this configuration itself contains the *cube* $\{x_1, x_1+x_2, x_1+x_3, x_1+x_2+x_3\}$.

We conclude the introduction with the statement of a remarkable generalization of Folkman's theorem.

Theorem 1.9 (Hindman's theorem)**.** *If $\mathbb{N}$ is finitely colored, then there exist an infinite sequence of distinct natural numbers $\{x_i\}$ and one color class that contains the* IP set
$$\left\{ \sum_{i \in I} x_i \,:\, I \subseteq \{1, \ldots, k\},\, 1 \leq |I| < \infty \right\}.$$

We shall unfortunately not prove this theorem in these notes, for a proof see [3].

2. Equivalence of "finite" and "infinite" Ramsey-type statements

All of the theorems above (with the exception of Hindman's theorem) are of the following general type:

Given $m \in \mathbb{N}$ and $\mathcal{F}$ a family of finite subsets of $\mathbb{N}$, then any m-coloring of $\mathbb{N}$ will have one color class that contains infinitely many representative from $\mathcal{F}$.

Our strategy to prove these theorems will be to establish "finite" variants of the following general type:

Given $m \in \mathbb{N}$ and $\mathcal{F}$ a family of finite subsets of $\mathbb{N}$, there exists a smallest integer $N_m(\mathcal{F})$, such that if $N \geq N_m(\mathcal{F})$, then any m-coloring of $\{1, \ldots, N\}$ will admit a monochromatic representative from $\mathcal{F}$.

It is clear that the "infinite" statements above are true whenever the "finite" statements are (for a fixed family of finite subsets $\mathcal{F}$), right? What is less obvious is that the converse is also true.

Theorem 2.1 (Compactness Principle)**.** *Let $m \in \mathbb{N}$ and $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$. Any m-coloring of $\mathbb{N}$ will have one color class that contains infinitely many representatives from $\mathcal{F}$ if and only if there exists a smallest integer $N_m(\mathcal{F})$, such that if $N \geq N_m(\mathcal{F})$, then any m-coloring of $\{1, \ldots, N\}$ will admit a monochromatic representative from $\mathcal{F}$.*

Proof. Sufficiency is clear, we will prove necessity. Let $m \in \mathbb{N}$ be fixed and assume that every m-coloring of $\mathbb{N}$ admits a monochromatic member of $\mathcal{F}$. We assume, for a contradiction, that for each $N \in \mathbb{N}$, there exists an m-coloring of $\{1, \ldots, N\}$

$$c_N : \{1, \ldots, N\} \to \{1, \ldots, m\}$$

that admits no monochromatic member of $\mathcal{F}$. We proceed by constructing a specific m-coloring, c, of $\mathbb{N}$.

Among $c_1(1), c_2(1), dots$ there must be a color that appears infinitely often. Call this color c^1 and let $\mathcal{C}_1$ denote the collection $\{c_N \,:\, c_N(1) = c^1\}$. Now within the set $\{c_N(2) \,:\, c_N \in \mathcal{C}_1\}$ there must exist a color, c^2, that appears infinitely often and we let $\mathcal{C}_2$ denote the collection $\{c_N \in \mathcal{C}_1 \,:\, c_N(2) = c^2\}$.

Iterating this process we obtain, for each $j \geq 2$, a color c^j such that the collection

$$\mathcal{C}_j = \{c_N \,:\, c_N(1) = c^1, \ldots, c_N(j) = c^j\}$$

is infinite. We define $c(j) := c^j$ for all $j \in \mathbb{N}$. By assumption, this m-coloring of $\mathbb{N}$ admits a monochromatic member of $\mathcal{F}$, say F. Let $j_0 = \max\{j \,:\, j \in F\}$. By construction, for every coloring $c_N \in \mathcal{C}_{j_0}$ we have that the values of $c_N(j)$ are equal for all $j \in F$. This contradicts our assumption that the c_N's avoid monochromatic members of $\mathcal{F}$, simply take $N = j_0$. □

3. Hilbert's theorem

Theorem 3.1 (Hilbert's theorem)**.** *Given $m, k \in \mathbb{N}$, there exists a smallest integer $H_m(k)$ such that if $N \geq H_m(k)$, then any m-coloring of $\{1, \ldots, N\}$ will admit a monochromatic* k-cube*, that is a monochromatic subset of the form*

$$\left\{ x_0 + \sum_{i \in I} x_i \,:\, I \subseteq \{1, \ldots, k\} \right\}.$$

Proof. Since a 1-cube is just a pair of natural numbers, it is clear that $H_m(1) = m + 1$. We proceed via induction on k, hence we let $k \geq 2$ and assume the existence of $H_m(k-1)$. We will now establish the result by demonstrating that $H_m(k) \leq mH_m(k-1)^k$. Towards this end we set $N = mH_m(k-1)^k$ and partition $\{1, \ldots, N\}$ into $mH_m(k-1)^{k-1}$ intervals, each of length $H_m(k-1)$. It then follows from the inductive hypothesis, that given any m-coloring of $\{1, \ldots, N\}$ each of these $mH_m(k-1)^{k-1}$ intervals will contain at least one monochromatic $(k-1)$-cube. But, up to translation, an interval of length $H_m(k-1)$ there are at most $(H_m(k-1) - 1)^{k-1} < H_m(k-1)^{k-1}$ such cubes. Since there are, by design, $mH_m(k-1)^{k-1}$ intervals, some two of these intervals will contain translations of the *same* $(k-1)$-cube in the *same* color. This completes the proof, since the union of a $(k-1)$-cube with a translate of itself is a k-cube. □

4. Ramsey's theorem and Schur's theorem

4.1. Ramsey's theorem.
We will establish the following pretty general version of Ramsey's theorem.

Theorem 4.1 (Ramsey's theorem)**.** *Given $m \in \mathbb{N}$ and $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 2}$, there exists a smallest integer $R_m(k_1, \ldots, k_m)$ such that if $N \geq R_m(k_1, \ldots, k_m)$ and $E = E_1 \cup \cdots \cup E_m$ is any partition of the edges of a complete graph with N vertices into m sets, then there exists $1 \leq j \leq m$ such that E_j contains a complete subgraph with k_j vertices.*

We now record the following special case (the case $k_1 = \cdots = k_m$ above) as a corollary as it is this result that we shall use in the proof of Schur's theorem below.

Corollary 4.2 (Ramsey's theorem (special case))**.** *Given $m \in \mathbb{N}$ and $k \in \mathbb{Z}_{\geq 2}$, there exists a smallest integer $R_m(k)$ such that if $N \geq R_m(k)$, then any m-coloring of the edges of a complete graph with N vertices will admit a monochromatic complete subgraph with k vertices.*

Note specifically that any m-coloring of a compete graph with at least $R_m(3)$ vertices will contain a monochromatic triangle. In Exercise 3 in Section 10 below you are asked to show that $R_m(3) \leq \lfloor em! \rfloor + 1$.

Proof of Ramsey's theorem. It suffices to prove the theorem for $m = 2$, see Exercise 2 in Section 10 below.

First note that $R_2(k_1, 2) = k_1$ for all $k_1 \geq 2$, and $R_2(2, k_2) = k_2$ for all $k_2 \geq 2$. We proceed via induction on the sum $k_1 + k_2$, having taken care of the case $k_1 + k_2 = 5$. Hence, we let $k_1 + k_2 \geq 6$ with $k_1, k_2 \geq 3$ and assume that both $R_2(k_1, k_2 - 1)$ and $R_2(k_1 - 1, k_2)$ exist. We will now establish the result by demonstrating that

$$R_2(k_1, k_2) \leq R_2(k_1 - 1, k_2) + R_2(k_1, k_2 - 1).$$

Let $N = R_2(k_1 - 1, k_2) + R_2(k_1, k_2 - 1)$, K_N denote a complete graph with N vertices and $E = E_1 \cup E_2$ be an arbitrary 2-coloring of the edges E of K_N. Pick on vertex from K_N and call it v. It follows from the pigeonhole principle that of the $N - 1$ edges from v to the other vertices of K_N, either at least $R_2(k_1 - 1, k_2)$ of them will be in E_1, or at least $R_2(k_1, k_2 - 1)$ of them will be in E_2. We will assume, with loss in generality, that the former occurs and let V denote the set of vertices connected to v by an edge in E_1. Since $|V| \geq R_2(k_1 - 1, k_2)$ it follows from the inductive hypothesis that the complete subgraph with vertex set V will contain either a complete subgraph with k_2 vertices whose edges are all in E_2, in which case we are done, or a complete subgraph with $k_1 - 1$ vertices whose edges are all in E_1, in which case we are also done since by connecting v to each vertex of this subgraph we will obtain a complete subgraph with k_1 vertices all of whose edges are in E_1. $\square$

4.2. Schur's theorem.

Theorem 4.3 (Schur's theorem)**.** *Given $m, k \in \mathbb{N}$, there exists a smallest integer $S_m(k)$ such that if $N \geq S_m(k)$, then any m-coloring of $\{1, \ldots, N\}$ will admit a monochromatic subset of the form*

$$\{x_1, \ldots, x_k, x_1 + \cdots + x_k\}.$$

The proof we present below gives the bound $S_m(k) \leq R_m(k+1) - 1$.

Proof of Schur's theorem. Let $N \geq R_m(k+1) - 1$, $c : \{1, \ldots, N\} \to \{1, \ldots, m\}$ be a fixed (but arbitrary) m-coloring of $\{1, \ldots, N\}$ and K_{N+1} denote a complete graph with $N + 1$ vertices that are labelled $1, \ldots, N + 1$. Using the given coloring c of $\{1, \ldots, N\}$ we now m-color the edges of K_{N+1} by assigning to the edge between the vertices i and j, the color $c(|j - i|)$. By Ramsey's theorem (in particular Corollary 4.2), the graph K_{N+1} must contain a monochromatic complete subgraph with $k + 1$ vertices. If we list the vertices of this monochromatic complete subgraph in order as $v_0 < v_1 < \cdots < v_k$, then it follows that the value of $c(v_j - v_i)$ will be equal for all $0 \leq i < j \leq k$. The claim then follows by setting $x_j := v_j - v_{j-1}$. $\square$

5. van der Waerden's theorem

Theorem 5.1 (van der Waerden's theorem). *Given $m, k \in \mathbb{N}$, there exists a smallest integer $W_m(k)$ such that if $N \geq W_m(k)$, then any m-coloring of $\{1, \ldots, N\}$ will admit a monochromatic arithmetic progression of length k, namely a set of the form $\{a, a+h, \ldots, a+(k-1)h\}$ with $h \neq 0$.*

We present the standard *color-focusing* proof of van der Waerden's theorem in Section 5.2 below, however before tackling this we strongly suggest that the reader try the following illuminating exercises.

5.1. Exercises.

1. Prove that $W_m(2) = m + 1$ for all $m \geq 1$.
2. Let k, m, N, x and d be positive integers. Prove that every m-coloring of $\{1, \ldots, N\}$ contains a monochromatic k-term arithmetic progression if and only if every m-coloring of
$$S = \{x, x+d, x+2d, \ldots, x+(N-1)d\}$$
contains a monochromatic k-term arithmetic progression.
3. Show that any two coloring of $\{1, \ldots, 325\}$ contains at least one monochromatic arithmetic progression of length three using the following steps (this strategy will be useful when we prove van der Waerden's theorem):
   (a) (i) Show that any 2-coloring of five consecutive natural numbers must contain a 3-term arithmetic progression whose first two elements are monochromatic.
   (ii) How many consecutive numbers are needed to ensure that any m-coloring contains a 3-term arithmetic progression whose first two elements are monochromatic?
   (b) Consider a 2-coloring of $\mathbb{N}$. How many consecutive blocks of the form
   $$\{x, x+1, x+2, x+3, x+4\}$$
   are needed to ensure a 3-term arithmetic progression of blocks where the first two blocks are identically colored?
   (c) Prove that any 2-coloring of $\{1, \ldots, 325\}$ must contain a monochromatic 3-term arithmetic progression.
4. A *fan of radius* 3 *and dimension* d *with base point* a is a d-tuple of the form
$$\{\{a, a+h_1, a+2h_1\}, \ldots, \{a, a+h_d, a+2h_d\}\}.$$
We say that a fan is *polychromatic* if its base point, a, and the *spokes*,
$$\{a+h_1, a+2h_1\}, \ldots, \{a+h_2, a+2h_2\},$$
are all monochromatic with distinct colors.

   Suppose that N is sufficiently large, $\{1, \ldots, N\}$ is 3-colored and contains no monochromatic 3-term arithmetic progressions.
   (a) Show that the coloring contains two identically colored blocks containing a fan of radius 3 and dimension 1.
   (b) Use part (a) to find two identically colored fans of radius 3 and dimension 2. Use this to show that the coloring must contain a 3-term arithmetic progression.
   (c) Can you now extend this argument to prove the existence of $W_m(3)$ for all $m \geq 1$?
5. Let $m \in \mathbb{N}$ and assume $W_m(3)$ exists.
   (a) Let $M \geq 1$ be an integer and 2-color $\{1, \ldots, MW_{2^M}(3)\}$. Prove that there exists a block, B, of M consecutive numbers such that $B, B+h$ and $B+2h$ are identically colored.
   (b) Fix $M \geq \frac{3}{2}W_2(3)$. Notice that any block of length M necessarily contains a 4-term arithmetic progression whose first three terms are monochromatic. Use this observation, together with part (a), to show that any two coloring of $\{1, \ldots, \frac{3}{2}MW_{2^M}(3)\}$ contains a monochromatic 4-term arithmetic progression.

5.2. **Proof of van der Waerden's theorem.** Some terminology and notation:

- We will use $a + [0, k-1]h$ to denote the k-term arithmetic progression $\{a, a+h, \dots, a+(k-1)h\}$.
- We define a *fan of radius k and dimension d with base point a* to be a d-tuple of the form
$$\{a + [0, k-1]h_1, \dots, a + [0, k-1]h_d\}.$$
- We call the progressions $a + [1, k-1]h_i$ the *spokes* of the fan.
- We will say that a fan is *polychromatic* if its base point a and the spokes, $a + [1, k-1]h_i$, $(1 \le i \le d)$ are all monochromatic with distinct colors.

Our proof will consist of two inductive steps. We will induct on the length of the progression k. To complete this induction, we will show that a coloring must either contain a monochromatic k-term arithmetic progression, or a polychromatic fan of radius k and dimension d by inducting on d. The key observation is that a polychromatic fan consisting of m colors, can only have $m-1$ spokes.

Proof of van der Waerden's theorem. The base case, when $k = 1$, is trivial. Assume that $k \ge 2$ and that there exists an N so that if $\{1, \dots, N\}$ is m-colored, then there is a monochromatic $(k-1)$-term arithmetic progression.

Claim 5.2. *For any $d \ge 1$ there exists M so that if a block of M consecutive numbers is m-colored, then either there is a monochromatic k-term arithmetic progression, or there exists a polychromatic fan of radius k and dimension d.*

The base case $d = 1$ follows from the existence of $W_m(k-1)$ claimed in the inductive hypothesis on progression length. Let $d \ge 2$ and assume the claim is true for $d-1$. Let M_1 and M_2 be large parameters. Consider M_2 consecutive blocks of M_1 consecutive integers. By the induction hypothesis, assuming M_1 is sufficiently large, either some block contains a k-term monochromatic progression (in which case we have proved van der Waerden's theorem) or each block must contain a polychromatic fan of radius k and dimension $d-1$. We will assume the latter.

Since there are m^{M_1} possible colorings of each block, as long as M_2 is sufficiently large, there must be an arithmetic progression of k blocks,
$$B, B+h, \dots, B+(k-1)h,$$
the last $k-1$ of which are identically colored (here we are using our first inductive hypothesis and the fact that we have assumed that there ar no monochromatic k-term arithmetic progressions). We note that it is sufficient to take $M_2 = 2W_{m^{M_1}}(k-1)$. Furthermore, the inductive hypothesis implies that the block $B+h$ must contain the elements of a polychromatic fan,
$$F + h = \{a + h + [0, k-1]h_1, \dots, a + h + [0, k-1]h_{d-1}\}$$
of radius k, dimension $d-1$ and base point a. Since the last $k-1$ blocks are identically colored, we now have a $k-1$ term progression of identically colored polychromatic fans
$$F + h, \dots, F + (k-1)h.$$
Then, the set
$$F \cup F + h \cup \cdots \cup F + (k-1)h$$
contains a polychromatic fan of radius k and dimension d. In particular, the fan
$$\big\{a + [0, k-1]h, a + [0, k-1](h + h_1), \dots, a + [0, k-1](h + h_{d-1})\big\}$$
is polychromatic. This completes the proof of our claim.

Let $d = m$. By the claim, as long as $N \ge M_1 M_2$ any m-coloring of $\{1, \dots, N\}$ must contain a k-term monochromatic arithmetic progression, or a polychromatic fan of radius k and dimension d. The latter case is impossible, which means that the coloring must contain a monochromatic k-term arithmetic progression. $\square$

6. Rado's single equation theorem

Theorem 6.1 (Rado's single equation theorem). *Let $m \in \mathbb{N}$ and $c_1, \ldots, c_k \in \mathbb{Z}$. If N is sufficiently large (depending only on $m, c_1, \ldots, c_k$), then any m-coloring of $\{1, \ldots, N\}$ will admit a monochromatic solution to the equation $c_1x_1 + \cdots + c_kx_k = 0$ if and only if some non-empty subset of $\{c_1, \ldots, c_k\}$ sums to zero.*

6.1. A strengthening of van der Waerden's theorem.

We first establish the following strengthening of van der Waerden's theorem (of which the case $s = 1$ is perhaps the most natural).

Theorem 6.2. *Given $m, k, s \in \mathbb{N}$, there exists a smallest integer $\widetilde{W}_m(k,s)$ such that if $N \geq \widetilde{W}_m(k,s)$, then any m-coloring of $\{1, \ldots, N\}$ will admit a monochromatic $(k+1)$-tuple of the form*

$$\{a, a+h, \ldots, a+(k-1)h, sh\}.$$

Proof. It is clear that $\widetilde{W}_1(k,s) = \max\{k,s\}$. We proceed via induction on m, hence we let $m \geq 2$ and assume the existence of $\widetilde{W}_{m-1}(k,s)$. We will now establish the result by demonstrating that

$$\widetilde{W}_m(k,s) \leq sW_m\left((k-1)\widetilde{W}_{m-1}(k,s)+1\right).$$

Let $N = sW_m((k-1)\widetilde{W}_{m-1}(k,s)+1)$. It then follows from van der Waerden's theorem that any m-coloring of $\{1, \ldots, N\}$ will necessarily contain a monochromatic arithmetic progression of length $(k-1)\widetilde{W}_{m-1}(k,s)+1$ within the first N/s natural numbers, that is a set of the form

$$\left\{a, a+h', \ldots, a+(k-1)\widetilde{W}_{m-1}(k,s)h'\right\} \subseteq \{1, \ldots, N/s\}$$

with $h' \neq 0$. Notice that if there exists a number sjh' that is the *same* color as this progression, with $1 \leq j \leq \widetilde{W}_{m-1}(k,s)$, then the result follows, with $h = jh'$. If this is not the case then the progression

$$\{sh', 2sh', \ldots, \widetilde{W}_{m-1}(k,s)sh'\}$$

must be $(m-1)$-colored, in which case the result follows immediately from the inductive hypothesis. In the final step we use the fact that every m-coloring $\{1, \ldots, N\}$ contains a monochromatic $(k+1)$-tuple of the form $\{a, a+h, \ldots, a+(k-1)h, sh\}$ if and only if for each $\lambda \in \mathbb{N}$ every m-coloring of $\{\lambda, 2\lambda, \ldots, N\lambda\}$ also contains a monochromatic $(k+1)$-tuple of the same form. □

6.2. Proof of Rado's single equation theorem.

Proof of Sufficiency. We assume, without loss in generality, that $c_1 + \cdots + c_{k'} = 0$ and fix a m-coloring of $\mathbb{N}$. If $k' = k$, we may take $x_1 = \cdots = x_k = 1$, hence we shall assume that $k' < k$ and $c_1 + \cdots + c_k \neq 0$.

Let $A = \gcd(c_1, \ldots, c_{k'})$. By the Euclidean algorithm, we can express A as a linear combination of $c_1, \ldots, c_{k'}$. If we let $B = c_{k'+1} + \cdots + c_k > 0$ and $s = A/\gcd(A,B)$, then it follows that there exists $t \in \mathbb{Z}$ such that

$$At + Bs = 0$$

since Bs is a multiple of A. Furthermore, we have $\lambda_1, \ldots, \lambda_{k'} \in \mathbb{Z}$ such that

$$c_1\lambda_1 + \cdots + c_{k'}\lambda_{k'} = At.$$

It is easy to then see that for any choices of a and h, the numbers

$$x_j = a + \lambda_j h$$

for $1 \leq j \leq k'$ and

$$x_j = sh$$

for $k'+1 \leq j \leq k$ will form solutions to our equation.

As a corollary of Theorem 6.2 it follows (almost immediately) that if $N \geq \widetilde{W}_m(2\Lambda+2, s)$, then given any m-coloring of $\{1, \ldots, N\}$ there must exist $a, h \in \mathbb{N}$ such that

$$\{a + \lambda h \,:\, |\lambda| \leq \Lambda\} \cup \{sh\}$$

is monochromatic. The result follows if we set $\Lambda = \max\{|\lambda_1|, \ldots |\lambda_{k'}|\}$. □

Proof of Necessity. Suppose that no subset of $\{c_1, \dots, c_k\}$ sums to zero. For some m, we construct a m-coloring of $\mathbb{Q} \setminus \{0\}$ with no monochromatic solution; this clearly forbids a monochromatic solution in $\mathbb{N}$. If $p > 0$ is prime, then each $q \in \mathbb{Q} \setminus \{0\}$ can be expressed uniquely as $q = p^j a/b$, with $p \nmid a$, $p \nmid b$ and $\gcd(a, b) = 1$. Let $c_p(q) = ab^{-1} \pmod p$; this defines a $(p-1)$-coloring of $\mathbb{Q}$. Note that $c_p(x) = c_p(y)$ implies that $c_p(\alpha x) = c_p(\alpha y)$ for all $\alpha \in \mathbb{Q} \setminus \{0\}$. Provided we choose p such that it does not divide any of the (finite number of) *non-zero* sums of subsets of $\{c_1, \dots, c_k\}$, it will suffice to show that our equation has no monochromatic solution with the c_p coloring.

Suppose to the contrary that $\{x_1, \dots, x_k\}$ forms a monochromatic solution. For all $\mu \in \mathbb{Q} \setminus \{0\}$, $\{\mu x_1, \dots, \mu x_k\}$ also forms a monochromatic solution. We may thus assume that $x_1, \dots, x_k \in \mathbb{N}$ with $\gcd(x_1, \dots, x_k) = 1$. We now reorder the set $\{x_1, \dots, x_k\}$ so that $x_1, \dots, x_{k'}$ are precisely the elements that are not divisible by p, we know that there is at least one such x_j. Reducing our equation modulo p; it becomes $c_1 x_1 + \cdots + c_{k'} x_{k'} \equiv 0 \pmod p$. Since the x_j's have the same color, their expressions as described above have the value $ab^{-1} \pmod p$. Dividing the congruence by this value, we obtain $c_1 + \cdots + c_{k'} \equiv 0 \pmod p$ which contradicts our choice of p. □

7. Folkman's theorem

Theorem 7.1 (Folkman's theorem)**.** *Given $m, k \in \mathbb{N}$, there exists a smallest integer $F_m(k)$ such that if $N \geq F_m(k)$, then any m-coloring of $\{1, \dots, N\}$ will admit a monochromatic subset of the form*

$$\left\{ \sum_{i \in I} x_i \,:\, I \subseteq \{1, \dots, k\},\, I \neq \emptyset \right\}$$

with $x_1, \dots, x_k$ distinct.

7.1. **A variant on Folkman's theorem.**

Instead of proving this result directly, we shall prove the following variant which allows an easier induction argument.

Theorem 7.2. *Given $m, k \in \mathbb{N}$, there exists a smallest integer $\widetilde{F}_m(k)$ such that if $N \geq \widetilde{F}_m(k)$, then any m-coloring of $\{1, \dots, N\}$ will admit a subset of the form*

$$\left\{ \sum_{i \in I} x_i \,:\, I \subseteq \{1, \dots, k\},\, I \neq \emptyset \right\}$$

with $x_1, \dots, x_k$ distinct that is weakly monochromatic *in the sense that the color of any element, $\sum_{i \in I} x_i$, depend only on the largest element of I.*

For example, the set

$$\left\{ \sum_{i \in I} x_i \,:\, I \subseteq \{1, 2, 3\},\, I \neq \emptyset \right\}$$

is *weakly monochromatic* precisely if its subsets

$$\{x_1\} \ \{x_2, x_1 + x_2\} \ \{x_3, x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3\}$$

are individually monochromatic.

Proof that Theorem 7.2 implies Folkman's theorem. Any weakly monochromatic set (on m-colors) of the form

$$\left\{ \sum_{i \in I} x_i \,:\, I \subseteq \{1, \dots, k'\},\, I \neq \emptyset \right\}$$

with $y_1, \dots, y_{k'}$ distinct, must contain a monochromatic set of the form

$$\left\{ \sum_{i \in I} x_i \,:\, I \subseteq \{1, \dots, k\},\, I \neq \emptyset \right\}$$

with $\{x_1, \dots, x_k\} \subseteq \{y_1, \dots, y_{k'}\}$ and $k \geq \lceil k'/m \rceil$. □

7.2. **Proof of Theorem 7.2.** It is clear that $\widetilde{F}_m(1) = 1$ for all $m \geq 1$. We proceed via induction on k, hence we let $k \geq 1$ and assume the existence of $\widetilde{F}_m(k)$. We will now establish the result by demonstrating that

$$\widetilde{F}_m(k+1) \leq W_m(\widetilde{F}_m(k)).$$

Let $N = W_m(\widetilde{F}_m(k) + 1)$. It then from van der Waerden's theorem that any m-coloring of $\{1, \ldots, N\}$ will necessarily contain a monochromatic arithmetic progression of length $\widetilde{F}_m(k) + 1$, that is a monochromatic set of the form

$$\{a, a+h, \ldots, a + \widetilde{F}_m(k)h\}$$

with $h \neq 0$. We now consider the arithmetic progression $\{h, 2h, \ldots, \widetilde{F}_m(k)h\}$. By the inductive hypothesis we know that this progression must contain a weakly monochromatic set of the form

$$\left\{ \sum_{i \in I} x_i \, : \, I \subseteq \{1, \ldots, k\}, \, I \neq \emptyset \right\}$$

with $x_1, \ldots, x_k$ distinct. If we add to this collection of x_j's the element $x_{k+1} := a$, then it is easy to see that the resulting set

$$\left\{ \sum_{i \in I} x_i \, : \, I \subseteq \{1, \ldots, k+1\}, \, I \neq \emptyset \right\}$$

will again be weakly monochromatic. □

8. Hales-Jewett theorem

Theorem 8.1 (Hales-Jewett theorem)**.** *Given $m, k \in \mathbb{N}$, there exists a smallest integer $HJ_m(k)$ such that if $d \geq W_m(k)$, then any m-coloring of $\{1, \ldots, k\}^d$ will admit a monochromatic* combinatorial line*, namely a set of the form*

$$\{a, a+v, \ldots, a+(k-1)v\}$$

with $a \in \{1, \ldots, k\}^d$ and $v \in \{0,1\}^d \setminus \{0, 0\}$.

8.1. **Shelah's proof.** It is clear that $HJ_m(1) = 1$. We proceed via induction on k, hence we let $k \geq 2$ and assume the existence of $HJ_m(k-1)$. We set $M = HJ_m(k-1)$ and write

$$[k]^d = [k]^{d_1} \times \cdots \times [k]^{d_M}$$

where $[k] = \{1, \ldots, k\}$, $d_1 = m^{(k-1)^{M-1}}$ and $d_i = m^{(k-1)^{M-i} + k^{d_1 + \cdots + d_{i-1}}}$ for $2 \leq i \leq M$. Our choice of d_M ensures that there exist

$$x = (\underbrace{k-1, \ldots, k-1}_{s}, k, \ldots, k) \in [k]^{d_M} \quad \text{and} \quad y = (\underbrace{k-1, \ldots, k-1}_{s'}, k, \ldots, k) \in [k]^{d_M}$$

with $0 \leq s < s' \leq d_M$ such that the affine subspaces

$$[k]^{d_1} \times \cdots \times [k]^{d_{M-1}} \times \{x\} \quad \text{and} \quad [k]^{d_1} \times \cdots \times [k]^{d_{M-1}} \times \{y\}$$

are colored identically. We now define the combinatorial line

$$L_M := \{(\underbrace{k-1, \ldots, k-1}_{s}, \underbrace{j, \ldots, j}_{s'-s}, k, \ldots, k) \, : \, 1 \leq j \leq k\} \subseteq [k]^{d_M}$$

and note that the elements on this line corresponding to $j = k-1$ and $j = k$ are the points x and y above.

We now induct down. Suppose that at the i-th step, with $1 \leq i \leq M-1$, we have

$$[k]^{d_1} \times \cdots \times [k]^{d_{i-1}} \times [k]^{d_i} \times L_{i+1} \times \cdots \times L_M \subseteq [k]^d$$

with the property that the color of any element in this space does not vary under any (single) coordinate changes from $k-1$ to k in $L_{i+1} \times \cdots \times L_M$. Since $d_i = m^{(k-1)^{M-i} + k^{d_1 + \cdots + d_{i-1}}}$, it follows as above that we can find a combinatorial line

$$L_i = \{(k-1, \ldots, k-1, j, \ldots, j, k, \ldots, k) \, : \, 1 \leq j \leq k\} \subseteq [k]^{d_i}$$

such that the color of $[k]^{d_1} \times \cdots \times [k]^{d_{i-1}} \times L_i \times L_{i+1} \times \cdots \times L_M$ is the same for $j = k-1$ and $j = k$ (in L_i).

This process leads to the construction of an affine subspace

$$L_1 \times \cdots \times L_M \subseteq [k]^d$$

of dimension M where each combinatorial line L_i takes the form

$$L_i = \{(k-1,\ldots,k-1,j,\ldots,j,k,\ldots,k) \,:\, 1 \leq j \leq k\} \subseteq [k]^{d_i}$$

and has the property that the color of $L_1 \times \cdots \times L_i \times \cdots \times L_M$ is the same for $j = k-1$ and $j = k$.

It then follows from the inductive hypothesis that if we identify in the natural way $L_1' \times \cdots \times L_M'$ with $\{1,\ldots,k-1\}^M$, where each L_i' is simply the first $k-1$ elements of L_i, namely

$$L_i' = \{(k-1,\ldots,k-1,j,\ldots,j,k,\ldots,k) \,:\, 1 \leq j \leq k-1\} \subseteq [k]^{d_i}$$

then $L_1' \times \cdots \times L_M'$ must contains a monochromatic combinatorial line. By construction it then follows immediately that $L_1 \times \cdots \times L_M \subseteq [k]^d$ also contains a monochromatic combinatorial line. $\square$

9. Remark on bounds for $W_m(k)$

We define the Ackermann hierarchy to be the sequence of functions $f_1, f_2, \cdots : \mathbb{N} \to \mathbb{N}$ defined by

$$f_1(x) = 2x \quad \text{(Double)}$$

and

$$f_{n+1}(x) = \underbrace{f_n \circ \cdots \circ f_n}_{x}(1)$$

for $n \geq 1$. Thus

$$f_2(x) = 2^x \quad \text{(Exponential)}$$

$$f_3(x) = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{x} \quad \text{(Tower)}$$

$$f_4(1) = 2,\ f_4(2) = 2^2 = 4,\ f_4(3) = 2^{2^{2^2}} = 65536,\ f_4(4) = \text{WOW!} \quad \text{(Wowzer)}$$

We say that a function $f : \mathbb{N} \to \mathbb{N}$ is of type n if there exist c and d with $f(cx) \leq f_n(x) \leq f(dx)$ for all x. Diagonalization allows for even faster growing functions. The Ackermann function $A : \mathbb{N} \to \mathbb{N}$ is defined by

$$A(x) = f_x(x).$$

Our first (color-focusing) proof of van der Waerden's theroem gives a bound on $W_2(k)$ which grows faster than f_n for all n – this is often a feature of such double inductions. The actual bounds obtained via this argument can be verified to be of Ackermann-type. Shelah's method of proof allows one to establish that

$$W_m(k) \leq f_4(k+m).$$

In 1998 Gowers established that

$$W_m(k) \leq 2^{2^{m^{2^{2^{k+9}}}}}$$

as a corollary of his seminal (Fourier analytic) proof of

Theorem 9.1 (Szemerédi's theorem)**.** *Given $0 < \delta \leq 1$ and $k \in \mathbb{N}$, there exists a smallest integer $N_0(\delta,k)$ such that if $N \geq N_0(\delta,k)$, then any subset $A \subseteq \{1,\ldots,N\}$ will density $|A|/N \geq \delta$ must contain an arithmetic progression of length k, namely a set of the form $\{a, a+h, \ldots, a+(k-1)h\}$ with $h \neq 0$.*

In particular Gowers established the bound

$$N_0(\delta,k) \leq 2^{2^{\delta^{-C_k}}}$$

where $C_k = 2^{2^{k+9}}$. It is easy of course an easy exercise to see that $W_m(k) \leq N_0(1/m,k)$ for all $m,k \in \mathbb{N}$.

10. Exercises

1. Explain why the fact that
$$\binom{k_1+k_2-2}{k_1-1} = \binom{k_1+k_2-3}{k_1-2} + \binom{k_1+k_2-3}{k_1-1}$$
implies that
$$R_2(k_1,k_2) \le \binom{k_1+k_2-2}{k_1-1}.$$
2. Prove that the existence of $R_2(k_1,k_2)$ implies the existence of $R_m(k_1,\dots,k_m)$ for all $m \ge 2$.
3. Prove that
$$R_m(3) \le \lfloor em! \rfloor + 1$$
by using induction on m. In particular show that if $R_m(3)$ exists, then
$$R_{m+1}(3) - 1 \le (m+1)(R_m(3)-1) + 1.$$
Prove that equality holds in the case when $m = 2, 3$.
4. Prove that for $k \ge 2$, $R_2(k) \ge 2^{k/2}$.

   Hint: Color the edges of the graph randomly.
5. Let $n \in \mathbb{Z}_{\ge 2}$ and p be prime. Prove that the congruence
$$x^n + y^n \equiv z^n \pmod p$$
has a solution in the integers, with p not dividing xyz, provided p is sufficiently large (depending on n only).

   Hint: Use the subgroup $\{x^n : x \in \mathbb{Z}_p^\times\}$ to partition $\mathbb{Z}_p^\times$.
6. Let $m, k \in \mathbb{N}$. Show that there exists a smallest integer $S'_m(k)$, such that any m-coloring of $\{1,\dots,N\}$, with $N \ge S'_m(k)$, will necessarily contain a monochromatic $(k+1)$-tuple of the form $\{x_1,\dots,x_k,x_1\cdots x_k\}$.
7. Let $m, k \in \mathbb{N}$. Show that there exists a smallest integer $\widetilde{S}_m(k)$, such that any m-coloring of $\{1,\dots,N\}$, with $N \ge \widetilde{S}_m(k)$, will necessarily contain a monochromatic $(k+1)$-tuple of the form $\{x_1,\dots,x_k,x_1\cdots x_k\}$ with $x_1,\dots,x_k$ distinct.
8. Show that there exists a 2-coloring of $\mathbb{N}$ that does not contain an infinitely long arithmetic progression.
9. Prove that the existence of $W_2(k)$ implies the existence of $W_m(k)$ for all $m \ge 2$.
10. Let $m, k \in \mathbb{N}$. Prove that there exists a constant $c(m,k)$ such that any m-coloring of $\{1,\dots,N\}$ will admit at least $\lfloor c(m,k)N^2 \rfloor$ monochromatic arithmetic progressions of length k in the same color.
11. Rado's theroem (full version) implies Folkman's theorem
12. (a) Hales-Jewett implies van der Waerden
    (b) Hales-Jewett implies Gallai

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