RADO'S SINGLE EQUATION THEOREM

REU SUMMER 2005

We present the special case of a single equation of Rado's theorem, on classifying partition regular homogeneous systems of linear equations. This will be done in a series of exercises. We start with a definition. Let $a_1, \ldots, a_k$ be non-zero integers. The equation

$$a_1 x_1 + \ldots + a_k x_k = 0$$

is called partition regular, if for every $r \in \mathbb{N}$ there exists an $N(r)$, such that if $[1, N(r)]$ is r-colored, then there exits a monochromatic solution $x_1, \ldots, x_k \in [1, N(r)]$.

1. Show that to every $r \in \mathbb{N}$ there is an $S(r)$, such that if $[1, S(r)]$ is r- colored, then there is a monochromatic triple $a, d, a+d$, by doing induction on r.

   Find a monochromatic AP: $\{a, a+d, \ldots, a+kd\}$, say it is Red. Notice that if any of the numbers ld where $1 \leq l \leq k$ is also Red then you're done. Otherwise use induction. What bound do you get for $S[r]$ in terms of $W[k, r]$?
2. Now consider the equation: $bx_1 + cx_2 - cx_3 = 0$. Look for monochromatic solutions in the form: $x_1 = l\,cd,\ x_2 = a,\ x_3 = a + l\,bd$, where $l = 1, 2, \ldots$.

   Choose a monochromatic AP: $\{a, a+bd, \ldots, a+kbd\}$, and examine the colors of the numbers $l\,cd$ $(l = 1, 2, \ldots)$.
3. Now consider the general homogeneous linear equation: $a_1 x_1 + \ldots + a_k x_k = 0$ $(a_i \neq 0,\ \forall\ i)$. Assume that there is a subset of coefficients, say $a_1, \ldots a_m$, such that $a_1 + \ldots + a_m = 0$.

   Show that finding a monochromatic solution of such an equation can be reduced to the case discussed in exercise 2. The idea is to look for solutions $x_1, \ldots, x_k$ which take only three distinct values.
4. Assume now that for any subset of coefficients: $a_{i_1} + \ldots + a_{i_m} \neq 0$. Show that if p is a prime not dividing any of the sums of the subsets of coefficients, the there is a p-coloring without monochromatic solution: $a_1 x_1 + \ldots + a_k x_k = 0$.

   This p- coloring is defined as follows. Let x_p denote the congruence class of x mod p. If p does not divide x, then let the color of x be x_p. If $p|x$ then write $x = p^s y$ where $p \nmid y$ and let the color of x be y_p.

   Show that there is no monochromatic solution for the above coloring.

Note that you proved that the equation: $a_1 x_1 + \ldots + a_k x_k = 0$ is partition regular, if and only if there is a subset of coefficients whose sum is zero. Can you generalize the condition to a pair homogeneous linear equations ?