REU Summer Research Report

Materials related to Gallai's Theorem

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Theorem 1 (Gallai). *Let k, r, d $\in \mathbb{N}$, and let F = $\{v_1, v_2, \ldots, v_k\}$ be a pattern $\in \mathbb{Z}^d$. Then there exists an N = N(k, d, r, $(v_1, v_2, \ldots, v_k)$) such that if $[1, N]^d \subseteqq \mathbb{Z}^d$ is colored with r colors, then there exists a monochromatic homothetic copy of the form*

$$x + tF = \{x, x + tv_1, \ldots, x + tv_k\}$$

where x is some element of $\mathbb{Z}^d$ and t is some element of $\mathbb{N}$.

We first need a few definitions to get an idea of the structure of the argument.

Definition 2. *An* $\mathbf{n-tuple}$ *of a pattern F = $\{v_1, v_2, \ldots, v_k\}$ with common base point x, is a set of the form A = $(x + t_1F) \cup \ldots \cup \ (x + t_nF)$. We say that A is* polychromatic*, if the sets: $\{x + t_iv_1, \ldots, x + t_iv_k\}$ are all monochromatic, but their colors are different.* NOTE *that the base point is not included in the sets.*

Definition 3. *A* $\mathtt{translate}$ *of a set A is an identical copy translated by some scalar multiple of one of the $v_i's$.*

Remark 4. *We'll begin with a crucial remark. If all the translates $A + tv_1, \ldots, A + tv_k$ are identically colored, then we are led to two cases:*

- *If the color of x is the same as one of the sets:*

$$\{x + (t + t_i)v_1, \ldots, x + (t + t_i)v_k\}$$

  then we have a monochromatic set consisting of:

$$\{x, x + (t + t_i)v_1, \ldots, x + (t + t_i)v_k\}$$

- *If not, then the set:*

$$\{x + tv_1, \ldots, x + tv_k\} \bigcup$$
$$\{x + (t + t_1)v_1, \ldots, x + (t + t_1)v_k\} \bigcup$$
$$\vdots$$
$$\{x + (t + t_n)v_1, \ldots, x + (t + t_n)v_k\}$$

  forms our polychromatic n + 1 tuple of the pattern F = $\{v_1, v_2, \ldots, v_k\}$.

Next, we need a necessary Lemma, which will aid in our discussion. ***Note***: The endpoint("tip") of each v_i can be viewed as an embedded design of size k - 1 within the larger design F. In other words, we will be led to a set of the form $x_i + t_iF'$ where

$$F' = \{(v_2 - v_1), (v_3 - v_1), \ldots, (v_k - v_1)\}.$$

Of course, once we have a set of the form $x_i + t_iF'$, we can return to our original design F from F' and one of the $x_i's$. This is will important in our lemma, as well as the following definition, which will ensure that our block will be large enough to completely contain F.

Definition 5. *Let M $\in \mathbb{N}$ be large enough such that $F \subseteqq [1, M]^d \subseteqq \mathbb{Z}^d$.*

Lemma 6. *Let r, k, d $\in \mathbb{N}$ be given and assume N(k - 1, d, r, F') exists for all patterns of size k - 1, where F' is as stated above. ie We have a monochromatic set of the form $x_i + t_i F'$. Then for all n $\in \mathbb{N}$, there exists N(k - 1, d, r, n, F') $\in \mathbb{N}$ such that if N $\geqq$ N(k - 1, d, r, n, F') and $[1, N]^d \subseteqq \mathbb{Z}^d$ is colored with r-colors, then there exists a monochromatic homothetic set of the form x + tF or a polychromatic n-tuple of F built as in our earlier* NOTE *above.*

Proof. Base Step (n = 1; trivial): The monochromatic set of the form $x_i + t_i F'$ will serve well as our monochromatic set of the form x + tF if x is colored the same as $(x_i, x_i + t_i(v_2 - v_1), \ldots, x_i + t_i(v_k - v_1))$, or as the polychromatic 1-tuple of the design F if x is not similarly colored.

Inductive Step: Assume the claim is true for n - 1. ie Assume that N_1 = N(k - 1, d, r, n - 1, F') exists. Then any "block" $\subseteqq \mathbb{Z}^d$ of size $(N_1)^d$ contains either a monochromatic set of the form x + tF or a polychromatic n - 1 tuple of F.

- In the first case, we are done.
- In the second case: Let $N_2 = max(2M\ ,\ 2N(k-1, d, (r^{N_1})^d, F'))$ If we consider $[1, N_2]^d \subseteqq \mathbb{Z}^d$ where each point is a block of size $(N_1)^d$, then we obtain a monochromatic set of blocks of the pattern F'. From our earlier note, we can denote these block as translates of the original pattern F and we'll call them $B + tv_1, \ldots, B + tv_k$. Since B is a block of size $(N_1)^d$, then it contains a polychromatic n - 1 tuple of F, which we'll denote by A $\subseteqq$ B. By definition, A is of the form $A = (x + t_1 F) \bigcup \ldots \bigcup\ (x + t_{n-1} F)$. Finally, following our initial crucial remark. If x is the same color as one of the sets:

  $$\{x + (t + t_i)v_1, \ldots, x + (t + t_i)v_k\}$$

  then we obtain our monochromatic set consisting of:

  $$\{x, x + (t + t_i)v_1, \ldots, x + (t + t_i)v_k\}$$

  or we obtain our polychromatic n-tuple of the pattern F. Now we are in a position to finish Gallai's Theorem.

□

Proof. Gallai's Theorem

- Base Step: $N(1, d, r, v_1) \geqq \ulcorner \|v_1\| \urcorner * r$ will suffice for designs of size 1.
- Inductive Step: Assume that N(k - 1, d, r, F') exists for each r. Then by the above Lemma $N(k-1, d, r, n, F')$ exists for each n. If we take n $\geqq$ r, then any r-coloring of $[1, N(k-1, d, r, r, F']^d \subseteqq \mathbb{Z}^d$ will contain a polychromatic r - tuple of F. The base point x must agree with one of $x + t_i F$ as there are only r colors. Thus we have our monochromatic $x + t_i F$.

□

Corollary 7 (Van der Waerden's Theorem)**.** *Let r, k $\in \mathbb{N}$ be given. Then there is a W(r, k)$\in \mathbb{N}$ such that if N $\geqq$ W(r, k) and $[1, n]$ is colored with r colors, then there is a monochromatic AP of length k.*

Proof. This is a special case of Gallai's thm where d = 1 and we denote each v_j = j. □