DISTANCES IN DENSE SUBSETS OF $\mathbb{Z}^d$

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Abstract. In [2] Katznelson and Weiss establish that all sufficiently large distances can always be attained between pairs of points from any given measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density. A second proof of this result, as well as a stronger "pinned variant", was given by Bourgain in [1] using Fourier analytic methods. In [5] the second author adapted Bourgain's Fourier analytic approach to established a result analogous to that of Katznelson and Weiss for subsets $\mathbb{Z}^d$ provided $d \geq 5$. In this article we establish an optimal strengthening of this discrete distance set result as well as the natural "pinned variant".

1. Introduction

Recall that *upper Banach density* δ^* is defined for $A \subseteq \mathbb{Z}^d$ by

$$\delta^*(A) = \lim_{N\to\infty} \sup_{x\in\mathbb{Z}^d} \frac{|A \cap (x + \{1, \ldots, N\}^d)|}{N^d}.$$

1.1. **Distance sets and existing results.** A result of Katznelson and Weiss [2] states that all sufficiently large distances can always be attained between pairs of points from any given measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density. Specifically, if A is a measurable subset of $\mathbb{R}^2$ of positive upper (Banach) density, then there exists $\lambda_0 = \lambda_0(A)$ such that the distance set

$$\operatorname{dist}(A) = \{|x - y| \,:\, x, y \in A\} \supseteq [\lambda_0, \infty).$$

This result was later established using Fourier analytic methods by Bourgain in [1]. Bourgain in fact also established a "pinned variant", namely that for any $\lambda_1 \geq \lambda_0$ there is a fixed $x \in A$ such that

$$\operatorname{dist}(A; x) = \{|x - y| \,:\, y \in A\} \supseteq [\lambda_0, \lambda_1].$$

In [5] the second author adapted Bourgain's Fourier analytic approach to established a result analogous to that of Katznelson and Weiss for subsets $\mathbb{Z}^d$, namely that if $A \subseteq \mathbb{Z}^d$ of positive upper (Banach) density and $d \geq 5$, then there exists $\lambda_0 = \lambda_0(A)$ and an integer q, depending only on the density of A, such that

$$\operatorname{dist}^2(A) = \{|x - y|^2 \,:\, x, y \in A\} \supseteq [\lambda_0, \infty) \cap q\mathbb{Z}.$$

Note that the fact that A could fall entirely into a fixed congruence class of some integer $1 \leq r \leq \delta^*(A)^{-1/d}$ ensures that q must be divisible by the least common multiple of all integers $1 \leq r \leq \delta^*(A)^{-1/d}$.

1.2. **New results.** In what follows we will denote the discrete sphere of radius $\sqrt{\lambda}$ by S_λ, namely

$$S_\lambda := \{x \in \mathbb{R}^d \,:\, |x|^2 = \lambda\} \cap \mathbb{Z}^d.$$

Our first result is the following optimal strengthening of the discrete distance set result from [5].

Theorem 1 (Optimal Unpinned Distances). *Let $\varepsilon > 0$ and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

There exist $q = q(\varepsilon)$ and $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any $\lambda \geq \lambda_0$ there exist $x \in A$ for which

$$\frac{|A \cap (x + qS_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon. \tag{1}$$

While the main result of this paper is the following (optimal) "pinned variant" of Theorem 1 above, in other words the (optimal) discrete analogue of Bourgain's pinned distances theorem.

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Theorem 2 (Optimal Pinned Distances). *Let $\varepsilon > 0$ and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

There exist $q = q(\varepsilon)$ and $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any given $\lambda_1 \geq \lambda_0$ there exists a fixed $x \in A$ such that

$$\frac{|A \cap (x + qS_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon \quad \text{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1. \tag{2}$$

2. Uniformly Distributed Sets

Definition 1 (Definition of q_η and η-uniform distribution). For any $\eta > 0$ we define

$$q_\eta := \operatorname{lcm}\{1 \leq q \leq C\eta^{-2}\} \tag{3}$$

with $C > 0$ a (sufficiently) large absolute constant and $A \subseteq \mathbb{Z}^d$ to be *η-uniformly distributed (modulo q_η)* if its relative upper Banach density on any "residue class" modulo q_η never exceeds $(1 + \eta^2)$ times its density on $\mathbb{Z}^d$, namely if

$$\delta^*(A \,|\, s + (q_\eta \mathbb{Z})^d) \leq (1 + \eta^2)\, \delta^*(A)$$

holds for all $s \in \{1, \ldots, q_\eta\}^d$.

Theorems 1 and 2 are immediate consequences, via an easy density increment argument, of the following analogous results for uniformly distributed sets.

Theorem 3 (Theorem 1 for Uniformly Distributed Sets). *Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^2$, and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

If A is η-uniformly distributed, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any $\lambda \geq \lambda_0$ one has

$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon \quad \text{for some } x \in A \tag{4}$$

Theorem 4 (Theorem 2 for Uniformly Distributed Sets). *Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^3$, and $A \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

If A is η-uniformly distributed, then there exist $\lambda_0 = \lambda_0(A, \varepsilon)$ such that for any given $\lambda_1 \geq \lambda_0$ there exists a fixed $x \in A$ such that

$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \delta^*(A) - \varepsilon \quad \text{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1. \tag{5}$$

3. Preliminaries

3.1. Fourier analysis on $\mathbb{Z}^d$.

If $f : \mathbb{Z}^d \to \mathbb{C}$ is a function for which

$$\sum_{x \in \mathbb{Z}^d} |f(x)| < \infty$$

we will say that $f \in \ell^1(\mathbb{Z}^d)$ and define

$$\|f\|_1 = \sum_{x \in \mathbb{Z}^d} |f(x)|.$$

For $f \in \ell^1$ we define its *Fourier transform* $\widehat{f} : \mathbb{T}^d \to \mathbb{C}$ by

$$\widehat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{-2\pi i x \cdot \xi}$$

noting that the summability assumption on f ensures that the series defining $\widehat{f}$ converges uniformly to a continuous function on the torus $\mathbb{T}^d$, which we will freely identify with the unit cube $[0,1)^d$ in $\mathbb{R}^d$.

Furthermore, Parseval's identity, namely that if $f, g \in \ell^1$ then

$$\langle f, g \rangle := \sum_{x \in \mathbb{Z}^d} f(x)\overline{g(x)} = \int_{\mathbb{T}^d} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi$$

is a simply and immediate consequence of the familiar orthogonality relation

$$\int_{\mathbb{T}^d} e^{2\pi i x \cdot \xi} d\xi = \begin{cases} 1 & \text{if } \ x = 0 \\ 0 & \text{if } \ x \neq 0 \end{cases} .$$

Defining the convolution of f and g to be

$$f*g(x)=\sum_{y\in\mathbb{Z}^d}f(x-y)g(y)$$

it follows that if $f,g\in\ell^1$ then $f*g\in\ell^1$ with

$$\|f*g\|_1\leq\|f\|_1\|g\|_1 \qquad \text{and} \qquad \widehat{f*g}=\widehat{f}\,\widehat{g}.$$

Finally, we recall following consequence of the Poisson Summation Formula, namely that if ψ is a Schwartz function on $\mathbb{R}^d$, then

$$\widehat{\psi}(\xi)=\sum_{y\in\mathbb{Z}^d}\widetilde{\psi}(\xi-y) \tag{6}$$

where

$$\widetilde{\psi}(\xi)=\int_{\mathbb{R}^d}\psi(x)e^{-2\pi ix\cdot\xi}\,dx \tag{7}$$

denotes the Fourier transform on $\mathbb{R}^d$ of ψ.

3.2. **Counting differences in S_λ.** Let $A\subseteq B_N$, where $B_N\subseteq\mathbb{Z}^d$ denotes some arbitrary translate of the cube $\{1,\dots,N\}^d$, and recall that we are denoting the discrete sphere of radius $\sqrt{\lambda}$ by S_λ, namely

$$S_\lambda:=\{x\in\mathbb{R}^d\,:\,|x|^2=\lambda\}\cap\mathbb{Z}^d.$$

It is easy to verify, using the properties of the Fourier transform discussed above, that

$$\sum_{x\in A}\frac{|A\cap(x+S_\lambda)|}{|S_\lambda|}=\langle 1_A,\mathcal{A}_\lambda(1_A)\rangle=\int|\widehat{1_A}(\xi)|^2\widehat{\sigma_\lambda}(\xi)\,d\xi \tag{8}$$

where $\mathcal{A}_\lambda(f)(x)$ denotes the spherical average

$$\mathcal{A}_\lambda(f)(x):=f*\sigma_\lambda(x)=\frac{1}{|S_\lambda|}\sum_{y\in S_\lambda}f(x-y). \tag{9}$$

3.3. **Exponential sum estimates.** In light of (8) we will naturally be interested estimates for the Fourier transform of the surface measure σ_λ, namely

$$\widehat{\sigma_\lambda}(\xi):=\frac{1}{|S_\lambda|}\sum_{x\in S_\lambda}e^{-2\pi ix\cdot\xi}. \tag{10}$$

It is clear that whenever $|\xi|^2\ll\lambda^{-1}$ there can be no cancellation in the exponential sum (10), in fact it is easy to verify that the same is also true whenever ξ is *close* to a rational point with *small* denominator. The following Proposition is a precise formulation of the fact that this is the only obstruction to cancellation.

Proposition 1 (Key exponential sum estimates, Proposition 1 in [5])**.** *Let $\eta>0$. If $\lambda\geq C\eta^{-4}$ and*

$$\xi\notin\left(q_\eta^{-1}\mathbb{Z}\right)^d+\{\xi\in\mathbb{R}^d\,:\,|\xi|^2\leq\eta^{-1}\lambda^{-1}\},$$

then

$$\Big|\frac{1}{|S_\lambda|}\sum_{x\in S_\lambda}e^{-2\pi ix\cdot\xi}\Big|\leq\eta.$$

3.4. **Smooth cutoff functions.** It will be convenient to introduce a smooth function $\psi_{q,L}$ whose Fourier transform (on $\mathbb{Z}^d$) will serve as a substitute for the characteristic function of the set

$$\mathfrak{M}_{q,L} = \left(q^{-1}\mathbb{Z}\right)^d + \{\xi \in \mathbb{R}^d \,:\, |\xi| \leq L^{-1}\}.$$

Towards this end, let $\psi : \mathbb{R}^d \to (0,\infty)$ be a Schwartz function satisfying

$$1 = \widetilde{\psi}(0) \geq \widetilde{\psi}(\xi) \geq 0 \qquad \text{and} \qquad \widetilde{\psi}(\xi) = 0 \ \text{ for } \ |\xi| > 1$$

where $\widetilde{\psi}$ denotes the Fourier transform (on $\mathbb{R}^d$) of ψ. For a given $q \in \mathbb{N}$ and $L \geq q$ we define

$$\psi_{q,L}(x) = \begin{cases} \left(\frac{q}{L}\right)^d \psi\left(\frac{x}{L}\right) & \text{if } \ x = (q\mathbb{Z})^d) \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

It follows from the Poisson summation formula that the Fourier transform (on $\mathbb{Z}^d$) of $\psi_{q,L}$ takes the form

$$\widehat{\psi}_{q,L}(\xi) = \sum_{\ell \in \mathbb{Z}^d} \widetilde{\psi}\left(L\left(\xi - \frac{\ell}{q}\right)\right) \tag{12}$$

and is supported on $\mathfrak{M}_{q,L}$.

3.5. **Properties of $\psi_{q,L}$ and $\widehat{\psi}_{q,L}$.** We first note that since $\widetilde{\psi}$ is compactly supported and $q \leq L$, it follows from (12) that

$$\sum_{x \in \mathbb{Z}^d} \psi_{q,L}(x) = \widehat{\psi}_{q,L}(0) = \sum_{\ell \in \mathbb{Z}^d} \widetilde{\psi}(\ell L/q) = \widetilde{\psi}(0) = 1.$$

We next make the simple but important observation that ψ may be chosen so that for any $\eta > 0$, the function $1 - \widehat{\psi}_{q,L}$ will be essentially supported on the complement of $\mathfrak{M}_{q,\eta^{-1}L}$ in the sense that

$$\left|1 - \widehat{\psi}_{q,L}(\xi)\right| \ll \eta \tag{13}$$

whenever $\xi \in \mathfrak{M}_{q,\eta^{-1}L}$.

Finally we record a precise formulation of the fact that $\psi_{q,L}$ is essentially supported on a box of size $\eta^{-1}L$ and is approximately constant on smaller scales.

Lemma 1. *Let $\eta > 0$ and $1 \leq q \leq L$, then*

$$\sum_{|x| \geq \eta^{-1}L} \psi_{q,L}(x) \ll \eta. \tag{14}$$

and

$$\|\chi_{q,L} * \psi_{q,L_1} - \psi_{q,L_1}\|_1 \ll \eta \tag{15}$$

whenever $L_1 \geq \eta^{-1}L$, where

$$\chi_{q,L}(x) = \begin{cases} \left(\frac{q}{L}\right)^d & \textit{if } \ x \in (q\mathbb{Z})^d \cap [-\frac{L}{2}, \frac{L}{2}]^d \\ 0 & \textit{otherwise} \end{cases}. \tag{16}$$

Proof. Estimate (14) is easily verified using the fact that ψ is a Schwartz function on $\mathbb{R}^d$ as

$$\sum_{|x| \geq \eta^{-1}L} \psi_{q,L}(x) = \left(\frac{q}{L}\right)^d \sum_{\substack{\ell \in \mathbb{Z}^d \\ |\ell| \geq \eta^{-1}L/q}} \psi(\ell q/L) \ll \left(\frac{q}{L}\right)^d \sum_{\substack{\ell \in \mathbb{Z}^d \\ |\ell| \geq \eta^{-1}L/q}} \left(1 + \frac{|\ell| q}{L}\right)^{-d-1} \ll \eta.$$

To verify estimate (15) we make use of the fact that both ψ and its derivative are rapidly decreasing, specifically

$$\begin{aligned} \|\chi_{q,L} * \psi_{q,L_1} - \psi_{q,L_1}\|_1 &\leq \left(\frac{q}{L}\right)^d \left(\frac{q}{L_1}\right)^d \sum_{x \in (q\mathbb{Z})^d} \sum_{y \in (q\mathbb{Z})^d \cap [-\frac{L}{2}, \frac{L}{2}]^d} \left|\psi\left(\frac{x-y}{L_1}\right) - \psi\left(\frac{x}{L_1}\right)\right| \\ &\leq \frac{L}{L_1} \left(\frac{q}{L_1}\right)^d \sum_{x \in (q\mathbb{Z})^d} \left(1 + \frac{|x|}{L_1}\right)^{-d-1} \leq \frac{L}{L_1}. \end{aligned}$$

□

4. Reducing Theorems 3 and 4 to Key Dichotomy Propositions

First a definition.

Definition 2 (Definition of (η, L)-uniform distribution)**.** Let N be a large positive integer and $B_N \subseteq \mathbb{Z}^d$ denotes some arbitrary translate of the cube $\{1,\dots,N\}^d$. For any $\eta > 0$ and positive integer L with the property that $q_\eta|L|N$ we define $A \subseteq B_N$ to be (η, L)-*uniformly distributed* if

$$\frac{|A \cap B_L \cap (s + (q_\eta\mathbb{Z})^d)|}{(L/q_\eta)^d} \leq (1+\eta^2)\frac{|A|}{N^d}$$

holds for all $s \in \{1,\dots,q_\eta\}^d$ and each sub-cube B_L in the partition of the original cube B_N into $(N/L)^d$ sub-cubes each of "sidelength" L.

4.1. Dichotomy Propositions.

As with the second author's approach in [5], itself adapted from [1], we will deduce Theorems 3 and 4 as consequences of the following quantitative finite versions.

Proposition 2 (Dichotomy for Theorem 3)**.** *Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^2$, and (L, N) be a pair of integers such that $q_\eta|L|N$. If $A \subseteq B_N \subseteq \mathbb{Z}^d$ with $d \geq 5$ is (η, L)-uniformly distributed, then for all integers λ satisfying $\eta^{-4}L^2 \leq \lambda \leq \eta^{11}N^2$ one of the following statements must hold:*

(i) *there exists $x \in A$ such that*
$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \frac{|A|}{N^d} - \varepsilon$$

(ii)
$$\frac{1}{|A|}\int_{\Omega_\lambda} |\widehat{1_A}(\xi)|^2\, d\xi \gg \varepsilon$$
where $\Omega_\lambda = \Omega_\lambda(\eta, q_\eta)$ denotes the set theoretic sum $\left(q_\eta^{-1}\mathbb{Z}\right)^d + \{\xi \in \mathbb{R}^d \,:\, \eta^2\lambda^{-1} \leq |\xi|^2 \leq \eta^{-2}\lambda^{-1}\}$.

Proposition 3 (Dichotomy for Theorem 4)**.** *Let $\varepsilon > 0$, $0 < \eta \ll \varepsilon^3$, and $A \subseteq B_N \subseteq \mathbb{Z}^d$ with $d \geq 5$.*

If A is (η, L)-uniformly distributed (this implicitly assumes that $q_\eta|L|N$), then for all integer pairs (λ_0, λ_1) that satisfy $\eta^{-4}L^2 \leq \lambda_0 \leq \lambda_1 \leq \eta^{11}N^2$ one of the following statements must hold:

(i) *there exists $x \in A$ with the property that one has*
$$\frac{|A \cap (x + S_\lambda)|}{|S_\lambda|} > \frac{|A|}{N^d} - \varepsilon \quad \textit{for all} \quad \lambda_0 \leq \lambda \leq \lambda_1$$

(ii)
$$\frac{1}{|A|}\int_{\Omega_{\lambda_0,\lambda_1}} |\widehat{1_A}(\xi)|^2\, d\xi \gg \varepsilon^2$$
where $\Omega_{\lambda_0,\lambda_1} = \Omega_{\lambda_0,\lambda_1}(\eta, q_\eta) = \left(q_\eta^{-1}\mathbb{Z}\right)^d + \{\xi \in \mathbb{R}^d \,:\, \eta^2\lambda_1^{-1} \leq |\xi|^2 \leq \eta^{-2}\lambda_0^{-1}\}$.

4.2. The Proof of Theorems 3 and 4.

We naturally start with a short Lemma relating our two notions of uniform distribution.

Lemma 2. *If $A \subseteq \mathbb{Z}^d$ is η-uniformly distributed. Then there exists a constant $L(A, \eta)$ such that for every positive integer $L \geq L(A, \eta)$ satisfying $q_\eta|L$ the following holds: There exist arbitrarily large positive integers N satisfying $L|N$ such that*

$$\text{(i)}\quad \frac{|A \cap B_N|}{N^d} \geq \delta^*(A) - \varepsilon/2 \quad \text{and} \quad \text{(ii)}\quad A \cap B_N \text{ is } (2\eta, L)\text{-uniformly distributed}$$

hold simultaneously for some cube B_N.

Proof. By our assumption there exists a positive integer $L(A, \eta)$ such that if $L' = L(A, \eta)$, then

$$\delta(A|(s + (q_\eta\mathbb{Z})^d) \cap B_{L'}) := \frac{|A \cap (s + (q_\eta\mathbb{Z})^d) \cap B_{L'}|}{|(s + (q_\eta\mathbb{Z})^d) \cap B_{L'}|} \leq (1 + 2\eta^2)\,\delta^*(A)$$

for any $s \in \{1,\dots,q_\eta\}^d$ and any cube $B_{L'}$ of size L'. Let $L \geq L'$ such that $q_\eta|L$.

Now choose any cube $B_{N'}$ of size $N' \gg \varepsilon^{-1}\eta^{-2}L$ such that $\delta(A|B_{N'}) \geq \delta^*(A)(1-\varepsilon\eta^2/20)$. Choosing $N' \leq N \leq N'+L$ one can ensure $L|N$ and $\delta(A|B_N) \geq \delta^*(A)(1-\varepsilon\eta^2/10)$, thus (i) holds (easily). To see (ii) note that $|(s+(q_\eta\mathbb{Z})^k)\cap B_L| = (L/q_\eta)^d$ and for $A' := A\cap B_N$ and any cube $B_L \subset B_N$ of size L we have

$$\frac{|A'\cap(s+(q_\eta\mathbb{Z})^d)\cap B_L|}{(L/q_\eta)^d} \leq (1+3\eta^2/2)\,\delta^*(A) \leq (1+2\eta^2)(1-\varepsilon\eta^2/10)^{-1}\frac{|A'|}{N^d} \leq (1+4\eta^2)\frac{|A'|}{N^d}. \qquad \square$$

4.2.1. *Proof that Proposition 2 implies Theorem 3.* Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^2$. Suppose that $A \subseteq \mathbb{Z}^d$ with $d \geq 5$ is an η-uniformly distributed set for which the conclusion of Theorem 3 fails to hold, namely that there exists arbitrarily large integers λ for which

$$\frac{|A\cap(x+S_\lambda)|}{|S_\lambda|} \leq \delta^*(A) - \varepsilon$$

for all $x \in A$. For a fixed integer $J \gg \varepsilon^{-1}$ we choose a sequence $\{\lambda^{(j)}\}_{j=1}^J$ of such λ's with the property that $\lambda^{(1)} \geq \eta^{-4}L^2$, $\lambda^{(j)} \leq \eta^4\lambda^{(j+1)}$ for $1 \leq j < J$, and $\lambda^{(J)} \leq \eta^{11}N^2$ with L and N satisfying the conclusion of Lemma 2. From Lemma 2 we obtain a set $A\cap B_N$, which we will abuse notation and denote by A.

An application Proposition 2 thus allows us to conclude that for this set one must have

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda^{(j)}}} |\widehat{1_A}(\xi)|^2\,d\xi \gg J\varepsilon > 1. \tag{17}$$

On the other hand it follows from the disjointness property of the sets $\Omega_{\lambda^{(j)}}$, which we guaranteed by our initial choice of sequence $\{\lambda^{(j)}\}$, and Plancherel that

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda^{(j)}}} |\widehat{1_A}(\xi)|^2\,d\xi \leq \frac{1}{|A|}\int_{\mathbb{T}^d} |\widehat{1_A}(\xi)|^2\,d\xi = 1 \tag{18}$$

giving a contradiction. $\square$

4.2.2. *Proof that Proposition 3 implies Theorem 4.* Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon^3$. Suppose that $A \subseteq \mathbb{Z}^d$ with $d \geq 5$ is an η-uniformly distributed set for which the conclusion of Theorem 4 fails to hold, namely that there exists arbitrarily large integer pairs (λ_0, λ_1) such that for all $x \in A$

$$\frac{|A\cap(x+S_\lambda)|}{|S_\lambda|} \leq \delta^*(A) - \varepsilon$$

for some $\lambda_0 \leq \lambda \leq \lambda_1$.

For a fixed integer $J \gg \varepsilon^{-2}$ we choose a sequence of such pairs $\{(\lambda_0^{(j)}, \lambda_1^{(j)}\}_{j=1}^J$ with the property that $\lambda_0^{(1)} \geq \eta^{-4}L^2$, $\lambda_0^{(j)} \leq \eta^4\lambda_1^{(j+1)}$ for $1 \leq j < J$, and $\lambda_1^{(J)} \leq \eta^{11}N^2$ with L and N satisfying the conclusion of Lemma 2. From Lemma 2 we obtain a set $A\cap B_N$, which we will abuse notation and denote by A.

An application Proposition 3 thus allows us to conclude that for this set one must have

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda_0^{(j)},\lambda_1^{(j)}}} |\widehat{1_A}(\xi)|^2\,d\xi \gg J\varepsilon^2 > 1. \tag{19}$$

On the other hand it follows from the disjointness property of the sets $\Omega_{\lambda_0^{(j)},\lambda_1^{(j)}}$, which we guaranteed by our initial choice of pair sequence $\{(\lambda_0^{(j)}, \lambda_1^{(j)})\}$, and Plancherel that

$$\sum_{j=1}^J \frac{1}{|A|}\int_{\Omega_{\lambda_0^{(j)},\lambda_1^{(j)}}} |\widehat{1_A}(\xi)|^2\,d\xi \leq \frac{1}{|A|}\int_{\mathbb{T}^d} |\widehat{1_A}(\xi)|^2\,d\xi = 1 \tag{20}$$

giving a contradiction. $\square$

5. Proof of Proposition 2

Let $f = 1_A$ and $\delta = |A|/N^d$. Suppose that $\eta^{-4}L^2 \le \lambda \le \eta^{11}N^2$ and that (i) does not hold, then

$$\langle f, \mathcal{A}_\lambda(f)\rangle \le \langle f, \delta - \varepsilon\rangle = (\delta - \varepsilon)|A|. \tag{21}$$

We now define

$$f_1 = f * \psi_{q_\eta, L_1} \quad \text{and} \quad f_2 = f * \psi_{q_\eta, L_2} \tag{22}$$

with $L_1 = \eta^{-1/2}\lambda^{1/2}$ and $L_2 = \eta\,\lambda^{1/2}$. Since

$$\big|\widehat{\psi}_{q_\eta, L_2}(\xi) - \widehat{\psi}_{q_\eta, L_1}(\xi)\big| \ll \eta^{1/2} \tag{23}$$

whenever $\xi \notin \Omega_\lambda = \mathfrak{M}_{q_\eta, L_2} \setminus \mathfrak{M}_{q_\eta, \eta^{-1/2}L_1}$, the proof of Proposition 2 is therefore reduced (via Parseval) to showing that if (21) holds, then

$$|\langle f, \mathcal{A}_\lambda(f_2 - f_1)\rangle| \gg \varepsilon |A|. \tag{24}$$

The observation that

$$|\langle f, \mathcal{A}_\lambda(f_2 - f_1)\rangle| \ge |\langle f, \mathcal{A}_\lambda(f_1)\rangle| - \langle f, \mathcal{A}_\lambda(f)\rangle - |\langle f, \mathcal{A}_\lambda(f - f_2)\rangle|$$

further reduces the entire argument to

Lemma 3 (Main term)**.** *If $f_1 := f * \psi_{q_\eta, L_1}$ with $L_1 = \eta^{-1/2}\lambda^{1/2}$, then $|\langle f, \mathcal{A}_\lambda(f_1)\rangle| \ge (\delta - C\eta^{1/2})|A|$.*

Lemma 4 (Error term)**.** *If $f_2 := f * \psi_{q_\eta, L_2}$ with $L_2 = \eta\,\lambda^{1/2}$, then $|\langle f, \mathcal{A}_\lambda(f - f_2)\rangle| \le \eta^{1/2}|A|$.*

Proof of Lemma 3. Since A is (η, L)-uniformly distributed it follows that $f * \chi_{q,L}(x) \le \delta(1 + \eta^2)$ for all $x \in \mathbb{Z}^d$.

As $L_1 \ge \eta^{-5/2}L$ and $\eta^{1/2} \ll \delta$ it further follows from the properties of $\psi_{q,L}$ discussed in Section 3.5 that

$$\begin{aligned} f_1(x) = f * \psi_{q,L_1}(x) &\le f * \chi_{q,L} * \psi_{q,L_1}(x) + |f * (\psi_{q,L_1} - \chi_{q,L} * \psi_{q,L_1})(x)| \\ &\le \delta(1 + \eta^2) + C\eta^{5/2} \le \delta(1 + C\eta^2). \end{aligned}$$

Let $N' = N + \eta^{-5/2}L_1$ and let $B_{N'}$ be a cube of size N' centered at the same point as B_N. As f is supported on B_N and $\eta^{1/2} \ll \delta$ we have

$$\sum_{x \in B_N} f_1(x) = \sum_{x \in \mathbb{Z}^k} f_1(x) - \sum_{x \notin B_{N'}} f_1(x) - \sum_{x \in B_{N'} \setminus B_N} f_1(x) \ge \delta(1 - C\eta^2)|B_N|. \tag{25}$$

Indeed, since $N \gg \eta^{-5}L_1$ we have

$$\frac{|B_{N'} \setminus B_N|}{|B_N|} \ll \left(\frac{N'}{N} - 1\right) \ll \eta^{-5/2}\frac{L_1}{N} \ll \eta^{5/2}$$

while from (14) we have

$$\sum_{x \notin B_{N'}} f_1(x) \le \sum_{|y| \gg \eta^{-5/2}L_1} \psi_{q,L_1}(y) \sum_x f(x - y) \le C\,\eta^{5/2}|B_N|.$$

We now define the set

$$E := \{x \in B_N;\ f_1(x) \le \delta - C\eta\}.$$

From estimate (25) it follows that

$$\delta(1 - C\eta^2)|B_N| \le \sum_{x \in E} f_1(x) + \sum_{x \in B_N \setminus E} f_1(x) \le |E|(\delta - C\eta) + (|B_N| - |E|)\delta(1 + C\eta^2)$$

and hence that $|E| \le C\eta\,\delta|B_N| = C\eta|A|$. Using the bound

$$f_1(x) \ge \delta - C\eta - 1_E(x)$$

for $x \in B_N$ it follows that

$$\langle f, \mathcal{A}_\lambda(f_1)\rangle \ge \langle f, \delta - C\eta\rangle - |\langle f, \mathcal{A}_\lambda(1_E)\rangle| \ge (\delta - C\eta)|A| - |\langle f, \mathcal{A}_\lambda(1_E)\rangle|.$$

The result follows via an application of Cauchy-Schwarz and the ℓ^2 boundedness of the operator $\mathcal{A}_\lambda$, namely that

$$\sum_{x\in\mathbb{Z}^d} |\mathcal{A}_\lambda(g)(x)|^2 \leq C \sum_{x\in\mathbb{Z}^d} |g(x)|^2$$

for any $g \in L^2$, which is an immediate consequence of Plancherel and the fact that $|\widehat{\sigma_\lambda}(\xi)| \leq 1$ for all $\xi \in \mathbb{T}^d$.

Indeed, with $g = 1_E$, we thus obtain

$$|\langle f, \mathcal{A}_\lambda(1_E)\rangle| \leq \Big(\sum_{x\in B_N} f(x)^2\Big)^{1/2} \Big(\sum_{x\in B_N} 1_E(x)\Big)^{1/2} \leq |A|^{1/2}|E|^{1/2} \leq \eta^{1/2}|A|. \qquad \square$$

Proof of Lemma 4. Note that

$$|\langle f, \mathcal{A}_\lambda(f-f_2)\rangle| \leq \int |\widehat{f}(\xi)|^2 |1-\widehat{\psi}_{q_\eta,L_2}(\xi)|\,|\widehat{\sigma_\lambda}(\xi)|\,d\xi.$$

Now Proposition 1 ensures that

$$|\widehat{\sigma_\lambda}(\xi)| \leq \eta$$

for all $\xi \notin \mathfrak{M}_{q_\eta,\eta^{-1/2}L_2}$ and ψ was constructed so that

$$\big|1-\widehat{\psi}_{q_\eta,L_2}(\xi)\big| \ll \eta^{1/2} \tag{26}$$

whenever $\xi \in \mathfrak{M}_{q_\eta,\eta^{-1/2}L_2}$. The result follows via Plancherel as $|\widehat{\sigma_\lambda}(\xi)| \leq 1$ for all $\xi \in \mathbb{T}^d$. $\square$

6. Proof of Proposition 3

Suppose that we have a pair (λ_0, λ_1) satisfying $\eta^{-4}L^2 \leq \lambda_0 \leq \lambda_1 \leq \eta^{11}N^2$, but for which (i) does not hold. It follows that there must exist $\lambda_0 \leq \lambda \leq \lambda_1$ such that

$$\langle f, \mathcal{A}_\lambda(f)\rangle \leq (\delta-\varepsilon)|A|$$

and hence that

$$\langle f, \mathcal{A}_*(1-f)\rangle \geq (1-\delta+\varepsilon/2)|A| \tag{27}$$

where $1 = 1_{B_N}$ and for any function $g : \mathbb{Z}^d \to \mathbb{C}$, $\mathcal{A}_*(g)$ denotes the *discrete spherical maximal function* defined by

$$\mathcal{A}_*(g)(x) := \sup_{\lambda_0\leq\lambda\leq\lambda_1} |\mathcal{A}_\lambda(g)(x)|.$$

Proposition 4 (ℓ^2-Boundedness of the Discrete Spherical Maximal Function [6])**.** *If $d \geq 5$, then*

$$\sum_{x\in\mathbb{Z}^d} |\mathcal{A}_*(g)(x)|^2 \leq C \sum_{x\in\mathbb{Z}^d} |g(x)|^2.$$

In light of Proposition 4, the proof of Proposition 3 reduces (via Cauchy-Schwarz and Plancherel) to showing that if (27) holds, then

$$|\langle f, \mathcal{A}_*(f_2-f_1)\rangle| \gg \varepsilon|A| \tag{28}$$

with $f_1 = f * \psi_{q_\eta,L_1}$ and $f_2 = f * \psi_{q_\eta,L_2}$, where now $L_1 = \eta^{-1/2}\lambda_1^{1/2}$ and $L_2 = \eta\,\lambda_0^{1/2}$.

Since

$$|\langle f, \mathcal{A}_*(f_2-f_1)\rangle| \geq |\langle f, \mathcal{A}_*(1-f)\rangle| - \langle f, \mathcal{A}_*(1-f_1)\rangle - |\langle f, \mathcal{A}_*(f-f_2)\rangle|$$

the whole argument reduces to

Lemma 5 (Main term)**.** *If $f_1 := f * \psi_{q_\eta,L_1}$ with $L_1 = \eta^{-1/2}\lambda_1^{1/2}$, then*

$$|\langle f, \mathcal{A}_*(1-f_1)\rangle| \leq (1-\delta+C\eta^{1/2})|A|.$$

Lemma 6 (Error term)**.** *If $f_2 := f * \psi_{q_\eta,L_2}$ with $L_2 = \eta\,\lambda_0^{1/2}$, then $|\langle f, \mathcal{A}_*(f-f_2)\rangle| \leq C\eta^{1/3}|A|$.*

6.1. **Proof of Lemma 5.** We use the lower bound

$$f_1(x) \geq \delta - C\eta - 1_E(x)$$

for $x \in B_N$ together with the bound $|E| \leq C\eta\,\delta|B_N|$ proved in Lemma 3. Then, as in the proof of Lemma 3, we obtain

$$|\langle f, \mathcal{A}_*(1-f_1)\rangle| \leq (1-\delta+C\eta)|A| + |\langle f, \mathcal{A}_*(1_E)\rangle|.$$

The result follows via an application of Cauchy-Schwarz and Proposition 4 since

$$|\langle f, \mathcal{A}_*(1_E)\rangle| \leq \Big(\sum_{x\in B_N} f(x)^2\Big)^{1/2}\Big(\sum_{x\in B_N} 1_E(x)\Big)^{1/2} \leq |A|^{1/2}|E|^{1/2} \leq \eta^{1/2}|A|.$$

□

6.2. **Proof of Lemma 6.** Note that

$$\mathcal{A}_*(f-f_2) = \sup_{\lambda_0\leq\lambda\leq\lambda_1} |(f-\psi_{q_\eta,L_2}*f)*\sigma_\lambda| = \sup_{\lambda_0\leq\lambda\leq\lambda_1} |f*(\sigma_\lambda - \sigma_\lambda*\psi_{q_\eta,L_2})| =: \mathcal{A}_{*,\eta}(f)$$

where the maximal operator $\mathcal{A}_{*,\eta}$ corresponds to the "mollified" multiplier $\widehat{\sigma}_{\lambda,\eta} := \widehat{\sigma}_\lambda(1-\widehat{\psi}_{q_\eta,L_2})$. Thus in order to prove the Lemma 6 it is suffices establish the following proposition.

Proposition 5 (ℓ^2-Decay of the "Mollified" Discrete Spherical Maximal Function)**.** *Let $f\in\ell^2$, then for any $\eta>0$ we have*

$$\sum_{x\in\mathbb{Z}^d} |\mathcal{A}_{*,\eta}(f)(x)|^2 \leq C\eta^{2/3}\sum_{x\in\mathbb{Z}^d}|f(x)|^2. \tag{29}$$

Proof of Proposition 5. We follow the proof of Proposition 4 given in [6]. For each $x\in\mathbb{Z}^d$ we now define

$$\widetilde{\mathcal{A}}_\lambda f(x) = \mathcal{A}_{\lambda^2} f(x) \tag{30}$$

noting that $\mathcal{A}_* f(x) = \sup_{\lambda_0^{1/2}\leq\lambda\leq\lambda_1^{1/2}} \widetilde{\mathcal{A}}_\lambda f(x) =: \widetilde{\mathcal{A}}_* f(x)$ and $\widetilde{\mathcal{A}}_{*,\eta}f(x) = \widetilde{\mathcal{A}}_*(f-f_2)(x)$.

We now recall the approximation to $\widetilde{\mathcal{A}}_\lambda$ given in Section 3 of [6] as a convolution operator $\mathcal{M}_\lambda$ acting on functions on $\mathbb{Z}^d$ of the form

$$\mathcal{M}_\lambda = c_d\sum_{q=1}^{\infty}\sum_{\substack{1\leq a\leq q\\(a,q)=1}} e^{-2\pi i\lambda a/q}\mathcal{M}_\lambda^{a/q} \tag{31}$$

where for each reduced fraction a/q the corresponding convolution operator $\mathcal{M}_\lambda^{a/q}$ has Fourier multiplier

$$m_\lambda^{a/q}(\xi) := \sum_{\ell\in\mathbb{Z}^k} G(a/q,\ell)\varphi_q(\xi-\ell/q)\widetilde{\sigma}_\lambda(\xi-\ell/q) \tag{32}$$

with $\varphi_q(\xi)=\varphi(q\xi)$ a standard smooth cut-off function, $G(a/q,l)$ a normalized Gauss sum, and $\widetilde{\sigma}_\lambda(\xi)=\widetilde{\sigma}(\lambda\xi)$ where $\widetilde{\sigma}(\xi)$ is the Fourier transform (on $\mathbb{R}^d$) of the measure on the unit sphere in $\mathbb{R}^d$ induced by Lebesgue measure and normalized to have total mass 1. By Proposition 4.1 in [6] we have

$$\Big\|\sup_{\Lambda\leq\lambda\leq2\Lambda}|\widetilde{\mathcal{A}}_\lambda(f)-\mathcal{M}_\lambda(f)|\Big\|_{\ell^2(\mathbb{Z}^d)} \leq C\Lambda^{-1/2}\|f\|_{\ell^2(\mathbb{Z}^d)} \tag{33}$$

provided $d\geq5$. Writing $\mathcal{M}_*(f) := \sup_{\lambda_0^{1/2}\leq\lambda\leq\lambda_1^{1/2}}|\mathcal{M}_\lambda(f)|$ and $\mathcal{M}_{*,\eta}(f) := \mathcal{M}_*(f-f_2)$, this implies

$$\|\widetilde{\mathcal{A}}_{*,\eta}(f)-\mathcal{M}_{*,\eta}(f)\|_{\ell^2} = \|\widetilde{\mathcal{A}}_*(f-f_2)-\mathcal{M}_*(f-f_2)\|_{\ell^2} \leq C\lambda_0^{-1/4}\|f-f_2\|_{\ell^2} \leq C\lambda_0^{-1/4}\|f\|_{\ell^2}. \tag{34}$$

Thus by choosing $\lambda_0 \gg \eta^{-4}$ matters reduce to showing (29) for the operator $\mathcal{M}_{*,\eta}$.

For a given reduced fraction a/q define the maximal operator

$$\mathcal{M}_*^{a/q}(f) := \sup_{\lambda_0^{1/2}\leq\lambda\leq\lambda_1^{1/2}}|\mathcal{M}_\lambda^{a/q}(f)|, \tag{35}$$

where $\mathcal{M}_\lambda^{a/q}$ is the convolution operator with multiplier $m_\lambda^{a/q}(\xi)$. It is proved in Lemma 3.1 of [6] that

$$\|\mathcal{M}_*^{a/q}(f)\|_{\ell^2} \leq Cq^{-d/2}\|f\|_{\ell^2}. \tag{36}$$

We will show here that if $q \leq C\eta^{-2/3}$, then

$$\|\mathcal{M}_*^{a/q}(f - f_2)\|_{\ell^2} \leq C\eta^{1/3} q^{-d/2}\|f\|_{\ell^2}. \tag{37}$$

Taking estimates (36) and (37) for granted, one obtains

$$\|\mathcal{M}_*(f - f_2)\|_{\ell^2} \ll \Big(\eta^{1/3} \sum_{1\leq q\leq C\eta^{-2/3}} q^{-d/2+1} + \sum_{q\geq C\eta^{-2/3}} q^{-d/2+1}\Big)\|f\|_{\ell^2} \ll \eta^{1/3}\|f\|_{\ell^2} \tag{38}$$

as required. It thus remains to prove (37).

Writing $\varphi_q(\xi) = \varphi'_q(\xi)\varphi_q(\xi)$, with a suitable smooth cut-off function φ', we can introduce the decomposition

$$m_\lambda^{a/q}(\xi) = \Big(\sum_{\ell\in\mathbb{Z}^k} G(a/q,\ell)\varphi'_q(\xi - \ell/q)\Big)\Big(\sum_{\ell\in\mathbb{Z}^k} \varphi_q(\xi - \ell/q)\widetilde{\sigma}(\xi - \ell/q)\Big) =: g^{a/q}(\xi)\, n_\lambda^q(\xi), \tag{39}$$

since for each ξ at most one term in each of the above sums is non-vanishing. Accordingly

$$\mathcal{M}_*^{a/q}(f - f_2) = G^{a/q}\ \mathcal{N}_*^q(f - f_2) \tag{40}$$

where the maximal operator $\mathcal{N}_*^q$ and the convolution operator $G_{a/q}$ correspond to the multipliers n_λ^q and $g^{a/q}$ respectively. Now by the standard Gauss sum estimate we have $|g^{a/q}(\xi)| \ll q^{-d/2}$ uniformly in ξ, hence

$$\|G^{a/q}\ \mathcal{N}_*^q(f - f_2)\|_{\ell^2} \ll q^{-d/2}\|\mathcal{N}_*^q(f - f_2)\|_{\ell^2}. \tag{41}$$

Thus by our choice $q_\eta := \mathrm{lcm}\{1 \leq q \leq C\eta^{-2}\}$ it remains to show that if q divides q_η then

$$\|\mathcal{N}_*^q(f - f_2)\|_{\ell^2} \ll \eta^{1/3}\|f\|_{\ell^2}. \tag{42}$$

As before we may write $\mathcal{N}_{*,\eta}^q(f) = \mathcal{N}_*^q(f - f_2)$, and note that this is a maximal operator with multiplier

$$n_\lambda^q(\xi)(1 - \widehat{\psi}_{q_\eta,L_2})(\xi) = \sum_{\ell\in\mathbb{Z}^d} \varphi_q(\xi - \ell/q)(1 - \widehat{\psi}_{q_\eta,L_2})(\xi - \ell/q)\widetilde{\sigma}_\lambda(\xi - \ell/q). \tag{43}$$

For a fixed q, the multiplier $\varphi_q(1 - \widehat{\psi}_{q_\eta,L_2})\widetilde{\sigma}_\lambda$ is supported on the cube $[-\frac{1}{2q}, \frac{1}{2q}]^d$ thus by Corollary 2.1 in [6]

$$\|\mathcal{N}_{*,\eta}^q\|_{\ell^2\to\ell^2} \leq C\,\|\widetilde{\mathcal{N}}_{*,\eta}^q\|_{L^2\to L^2}$$

where the latter is the maximal operator corresponding to the multipliers $\varphi_q(1 - \widehat{\psi}_{q_\eta,L_2})\widetilde{\sigma}_\lambda$, for $\lambda_0^{1/2} \leq \lambda \leq \lambda_1^{1/2}$, acting on $L^2(\mathbb{R}^d)$. By the definition of the function $\psi_{q,L}$

$$|1 - \widehat{\psi}_{q_\eta,L_2}(\xi)| \ll \min\{1, L_2|\xi|\},$$

thus from Theorem 6.1 (with $j = 1$) in [3] we obtain

$$\|\widetilde{\mathcal{N}}_{*,\eta}^q\|_{L^2\to L^2} \ll \left(\frac{L_2}{\lambda_0^{1/2}}\right)^{1/3} \ll \eta^{1/3}$$

which establishes (42) and completes the proof. □

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