Math 8100 Assignment 5
Repeated Integration

Due date: Friday the 18th of October 2019

1. Prove that if $\{a_{jk}\}_{(j,k)\in\mathbb{N}\times\mathbb{N}}$ is a "double sequence" with $a_{jk}\geq 0$ for all $(j,k)\in\mathbb{N}\times\mathbb{N}$, then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}=\sup\Big\{\sum_{(j,k)\in B}a_{jk}\,:\,B\text{ is a finite subset of }\mathbb{N}\times\mathbb{N}\Big\}$$

and deduce from this that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{jk}.$$

This conclusion holds more generally provided $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|<\infty$, *see Theorem 8.3 in "Baby Rudin".*

2. Let $f\in L^1([0,1])$, and for each $x\in[0,1]$ define

$$g(x)=\int_x^1\frac{f(t)}{t}\,dt.$$

Show that $g\in L^1([0,1])$ and that

$$\int_0^1 g(x)\,dx=\int_0^1 f(x)\,dx.$$

3. Carefully prove that if we define

$$f(x,y):=\begin{cases}\dfrac{x^{1/3}}{(1+xy)^{3/2}} & \text{if } 0\leq x\leq y\\ 0 & \text{otherwise}\end{cases}$$

for each $(x,y)\in\mathbb{R}^2$, then f defines a function in $L^1(\mathbb{R}^2)$.

4. Let $A,B\subseteq\mathbb{R}^n$ be bounded measurable sets with positive Lebesgue measure. For each $t\in\mathbb{R}^n$ define the function

$$g(t)=m\left(A\cap(t-B)\right)$$

where $t-B=\{t-b\,:\,b\in B\}$.

   (a) Prove that g is a continuous function and

$$\int_{\mathbb{R}^n}g(t)\,dt=m(A)\,m(B).$$

   (b) Conclude that the sumset

$$A+B=\{a+b\,:\,a\in A\text{ and }b\in B\}$$

   contains a non-empty open subset of $\mathbb{R}^n$.

5. Let $f,g\in L^1([0,1])$ and for each $0\leq x\leq 1$ define

$$F(x):=\int_0^x f(y)\,dy\quad\text{and}\quad G(x):=\int_0^x g(y)\,dy.$$

Prove that

$$\int_0^1 F(x)g(x)\,dx=F(1)G(1)-\int_0^1 f(x)G(x)\,dx.$$

6. Let $f \in L^1(\mathbb{R})$. For any $h > 0$ we define

$$A_h(f)(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y)\,dy$$

(a) Prove that for all $h > 0$,

$$\int_{\mathbb{R}} |A_h(f)(x)|\,dx \leq \int_{\mathbb{R}} |f(x)|\,dx.$$

(b) Prove that

$$\lim_{h \to 0^+} \int_{\mathbb{R}} |A_h(f)(x) - f(x)|\,dx = 0.$$

One can in fact show that $\lim_{h\to 0^+} A_h(f) = f$ *almost everywhere. This result is actually equivalent to the Lebesgue Density Theorem in* $\mathbb{R}$ *and we will establish this later in the course.*

Extra Challenge Problems

Not to be handed in with the assignment

1. (a) Prove that

$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx = \infty.$$

(b) By considering the iterated integral

$$\int_0^\infty \left(\int_0^\infty x e^{-xy}(1 - \cos y)\,dy \right) dx$$

show (with justification) that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

2. Suppose that F is a closed subset of $\mathbb{R}$ whose complement has finite measure. Let $\delta(x)$ denote the distance from x to F, namely

$$\delta(x) = d(x, F) = \inf\{|x - y| : y \in F\}$$

and

$$I_F(x) = \int_{-\infty}^{\infty} \frac{\delta(y)}{|x - y|^2}\,dy.$$

(a) Prove that δ is continuous, by showing that it satisfies the Lipschitz condition $|\delta(x) - \delta(y)| \leq |x - y|$.

(b) Show that $I_F(x) = \infty$ if $x \notin F$.

(c) Show that $I_F(x) < \infty$ for a.e. $x \in F$, by showing that $\int_F I_F(x)\,dx < \infty$.