Math 8100 Assignment 8
Basic Function Spaces

Due date: Friday the 16th of November 2018

1. Prove the following basic properties of $L^\infty = L^\infty(X)$, where X is a measurable subset of $\mathbb{R}^n$:
   (a) $\|\cdot\|_\infty$ is a norm on L^∞ and when equipped with this norm L^∞ is a Banach space.
   (b) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathbb{R}^n$ such that $m(E^c) = 0$ and $f_n \to f$ uniformly on E.
   (c) Simple functions are dense in L^∞, but continuous functions with compact support are not.

   Recall that if $X \subseteq \mathbb{R}^n$ is measurable and f is a measurable function on X, then we define
   $$\|f\|_\infty = \inf\{a \geq 0 : m(\{x \in X : |f(x)| > a\}) = 0\},$$
   with the convention that $\inf \emptyset = \infty$, *and*
   $$L^\infty = L^\infty(X) = \{f : X \to \mathbb{C} \textit{ measuarable} : \|f\|_\infty < \infty\},$$
   with the usual convention that two functions that are equal a.e. define the same element of L^∞. Thus $f \in L^\infty$ if and only if there is a bounded function g such that $f = g$ almost everywhere; we can take $g = f\chi_E$ where $E = \{x : |f(x)| \leq \|f\|_\infty\}$.

2. Let $X \subseteq \mathbb{R}^n$ be measurable.
   (a) i. Prove that if $m(X) < \infty$, then
   $$L^\infty(X) \subset L^2(X) \subset L^1(X) \tag{1}$$
   with strict inclusion in each case, and that for any measurable $f : X \to \mathbb{C}$ one in fact has
   $$\|f\|_{L^1(X)} \leq m(X)^{1/2}\|f\|_{L^2(X)} \leq m(X)\|f\|_{L^\infty(X)}.$$
   ii. Give examples to show that no such result of the form (1) can hold if one drops the assumption that $m(x) < \infty$. Prove, furthermore, that if $L^2(X) \subseteq L^1(X)$, then $m(X) < \infty$.
   (b) Prove that
   $$\underbrace{L^1(X) \cap L^\infty(X) \subset L^2(X)}_{(\star)} \subset L^1(X) + L^\infty(X)$$
   and that in addition to $(\star)$ one in fact has
   $$\|f\|_{L^2(X)} \leq \|f\|_{L^1(X)}^{1/2}\|f\|_{L^\infty(X)}^{1/2}$$
   for any measurable function $f : X \to \mathbb{C}$.

3. Prove that
   $$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$$
   with strict inclusion in each case, and that for any sequence $a = \{a_j\}_{j\in\mathbb{Z}}$ of complex numbers one in fact has
   $$\|a\|_{\ell^\infty(\mathbb{Z})} \leq \|a\|_{\ell^2(\mathbb{Z})} \leq \|a\|_{\ell^1(\mathbb{Z})}.$$
   Recall that for $p = 1, 2, \infty$ we define
   $$\ell^p(\mathbb{Z}) = \{a = \{a_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{C} : \|a\|_{\ell^p(\mathbb{Z})} < \infty\}$$
   where
   $$\|a\|_{\ell^1(\mathbb{Z})} = \sum_{j=-\infty}^{\infty} |a_j|, \quad \|a\|_{\ell^2(\mathbb{Z})} = \Big(\sum_{j=-\infty}^{\infty} |a_j|^2\Big)^{1/2}, \textit{ and } \|a\|_{\ell^\infty(\mathbb{Z})} = \sup_j |a_j|.$$

4. Let H be a Hilbert space with orthonormal basis $\{u_n\}_{n=1}^{\infty}$.

   (a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Prove that

   $$\sum_{n=1}^{\infty} a_n u_n \text{ converges in } H \iff \sum_{n=1}^{\infty} |a_n|^2 < \infty,$$

   and moreover that if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, then $\Big\|\sum_{n=1}^{\infty} a_n u_n\Big\| = \Big(\sum_{n=1}^{\infty} |a_n|^2\Big)^{1/2}$.

   (b) i. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1}$ for all $n \in \mathbb{N}$? If L exists, find its norm.

   ii. Is there a continuous linear functional L on H such that $L(u_n) = n^{-1/2}$ for all $n \in \mathbb{N}$? If L exists, find its norm.

5. For each $1 \le p \le \infty$, define $\Lambda_p : L^p([0,1]) \to \mathbb{R}$ by

   $$\Lambda_p(f) = \int_0^1 x^2 f(x)\, dx.$$

   Explain why Λ_p is a continuous linear functional and compute its norm (in terms of p).

Extra Practice Problems

Not to be handed in with the assignment

1. Let $C([0,1])$ denote the space of all continuous $\mathbb{C}$-valued functions on $[0,1]$.

   (a) Prove that $C([0,1])$ is complete under the uniform norm $\|f\|_u := \max_{x\in[0,1]} |f(x)|$.

   (b) Prove that $C([0,1])$ is <u>not</u> complete under either the $L^1([0,1])$ or $L^2([0,1])$ norms.

2. Let f and g be two non-negative Lebesgue measurable functions on $[0,\infty)$. Suppose that

   $$A := \int_0^{\infty} f(y)\, y^{-1/2} dy < \infty \qquad \text{and} \qquad B := \left(\int_0^{\infty} |g(y)|^2 dy\right)^{1/2} < \infty$$

   Prove that

   $$\int_0^{\infty} \left(\int_0^x f(y)\, dy\right) \frac{g(x)}{x}\, dx \le AB$$

3. Let $\{f_k\}$ be any sequence of functions in $L^2([0,1])$ satisfying $\|f_k\|_2 \le 1$ for all $k \in \mathbb{N}$.

   (a) i. Prove that if $f_k \to f$ either a.e. on $[0,1]$ or in $L^1([0,1])$, then $f \in L^2([0,1])$ with $\|f\|_2 \le 1$.

   ii. Do either of the above hypotheses guarantee that $f_k \to f$ in $L^2([0,1])$?

   (b) Prove that if $f_k \to f$ a.e. on $[0,1]$, then this in fact implies that $f_k \to f$ in $L^1([0,1])$.

4. Let $1 \le p \le \infty$. Prove that if $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $L^p(\mathbb{R}^n)$ with the property that

   $$\sum_{k=1}^{\infty} \|f_k\|_p < \infty,$$

   then $\sum f_k$ converges almost everywhere to an $L^p(\mathbb{R}^n)$ function with

   $$\Big\|\sum_{k=1}^{\infty} f_k\Big\|_p \le \sum_{k=1}^{\infty} \|f_k\|_p.$$

Extra Challenge Problems on Fourier Series
Not to be handed in with the assignment

Recall that if $f \in L^1(\mathbb{T}) := \{f \in L^1([0,1]) \,:\, f(0) = f(1)\}$, then the *Nth partial sum of the Fourier series of* f, is defined be

$$S_N f(x) = \sum_{|n| \leq N} \widehat{f}(n) e^{2\pi i n x}$$

where

$$\widehat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx,$$

for each $n \in \mathbb{Z}$.

1. (a) Prove that if $f \in L^2(\mathbb{T})$ and $\{\widehat{f}(n)\}_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$, then $S_N f$ converges uniformly to f for almost every $x \in [0,1]$ and for every $x \in [0,1]$ if one makes the additional assumption that $f \in C(\mathbb{T})$, namely 1-periodic and continuous.

   (b) i. Prove that if $f \in C^1(\mathbb{T})$, then $S_N f$ converges uniformly to f.
   Hint: Use Cauchy-Schwarz and Parseval for f'.

   ii. Prove that if $f \in C(\mathbb{T})$ and $f' \in L^2(\mathbb{T})$, then $S_N f$ converges uniformly to f.

Both results in part (b) above in fact follow from the following deeper result:

Theorem 1 (Dini's Criterion)**.** *If, for some $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, there exists $\delta > 0$ such that*

$$\int_{|t| \leq \delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty \tag{2}$$

then $S_N f(x)$ converges to $f(x)$.

Note that if f is *Hölder continuous* at x, namely $|f(x+t) - f(x)| \leq C|t|^a$ for some $a > 0$, then f satisfies (2) for some $\delta > 0$. But, continuous functions need not satisfy (2) for any $\delta > 0$, in fact:

Theorem 2 (Du Bois-Reymond)**.** *There exist $f \in C(\mathbb{T})$ whose Fourier series diverges at a point.*

It is straightforward to see that one can re-express the Nth partial sums as follows:

$$S_N f(x) = f * D_N(x) := \int_0^1 f(y) D_N(x-y)\, dy$$

where

$$D_N(x) := \sum_{|n| \leq N} e^{2\pi i n x} = \frac{\sin((2N+1)\pi x)}{\sin \pi x} \qquad \text{(Dirichlet kernel).}$$

We shall now consider the *Cesàro means* of the $S_N f$, namely

$$\sigma_N f := \frac{1}{N} \sum_{n=0}^{N-1} S_n f = f * F_N$$

where

$$F_N(x) := \frac{1}{N} \sum_{n=0}^{N-1} D_n(x) = \frac{1}{N} \Big(\frac{\sin(N\pi x)}{\sin \pi x} \Big)^2 \qquad \text{(Fejér kernel).}$$

2. (a) Verify that the Fejér kernel satisfies the following basic properties:

   i. $0 \le F_N(x) \le C\dfrac{1}{N}\min\Big\{N^2, \dfrac{1}{|x|^2}\Big\}$ for some constant $C > 0$ and all $x \in [0,1]$,

   ii. $\displaystyle\int_0^1 F_N(x)\,dx = 1$,

   iii. $\displaystyle\lim_{N\to\infty} \int_{\delta \le |x| \le \frac{1}{2}} F_N(x)\,dx = 0$ for any choice of $\delta > 0$.

   [Note also that $\widehat{F_N}(n) = \max\Big\{1 - \dfrac{|n|}{N}, 0\Big\}$ for all $n \in \mathbb{Z}$.]

   (b) Use the *approximation to the identity*-type properties above to prove the following

   Theorem 3 (Fejér's Theorem)**.** *Let* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$*.*

   (i) (Classical version) *If* $f \in C(\mathbb{T})$*, then* $\sigma_N f \to f$ *uniformly on* $\mathbb{T}$ *as* $N \to \infty$*.*

   (ii) (L^1-version) *If* $f \in L^1(\mathbb{T})$*, then* $\sigma_N f \to f$ *in* $L^1(\mathbb{T})$ *as* $N \to \infty$*.*

   [It is also true that if $f \in L^p(\mathbb{T})$ with $1 \le p < \infty$, then $\sigma_N f \to f$ in $L^p(\mathbb{T})$ as $N \to \infty$.]

   (c) Verify that Theorem 3 gives a new proof that *Trigonometric polynomials are dense in both* $C(\mathbb{T})$ *and in* $L^1(\mathbb{T})$, and that Theorem 1 (ii) in particular has the following important (new) consequence:

   Corollary 1.

   If $f \in L^1(\mathbb{T})$ *and* $\widehat{f}(n) = 0$ *for all* $n \in \mathbb{Z}$*, then* $f(x) = 0$ *for almost every* $x \in \mathbb{T}$*.*

3. Use Corollary 1 above to prove the following strengthening of Question 1 (a) above:

   Theorem 4 (Periodic analogue of the Fourier inversion formula)**.**

   If $f \in L^1(\mathbb{T})$ *and* $\{\widehat{f}(n)\} \in \ell^1(\mathbb{Z})$*, then* $S_N f(x) \to f(x)$ *for almost every* $x \in \mathbb{T}$ *as* $N \to \infty$*.*

4. (a) i. Prove that if f is continuous and periodic with period 1, and α is irrational, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(x)\,dx.$$

   Hint: Use the "Periodic Weierstrass Approximation Theorem".

   ii. Conclude that if α is irrational, then the sequence of fractional parts $\langle\alpha\rangle, \langle 2\alpha\rangle, \langle 3\alpha\rangle, \ldots$, where $\langle x\rangle = x - \lfloor x\rfloor$, is equidistributed in $[0,1)$, that is for every interval $(a,b) \subset [0,1)$,

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N \,:\, \langle n\alpha\rangle \in (a,b)\}}{N} = b - a.$$

   (b) Prove that following more general criterion:

   Theorem 5 (Weyl's Criterion)**.** *The following assertions concerning a given sequence* $\{\xi_n\}$ *in* $[0,1)$ *are equivalent:*

   (i) *The sequence* $\{\xi_n\}$ *is equidistributed;*

   (ii) *For each integer* $k \ne 0$*,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0;$$

   (iii) *For any (Riemann) integrable function* f *on* $[0,1]$ *that is periodic with period 1*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\xi_n) = \int_0^1 f(x)\,dx.$$