Math 8100 Assignment 3
Lebesgue measurable sets and functions

Due date: Wednesday the 19th of September 2018

1. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ with $m(E) < \infty$ and $\varepsilon > 0$. Show that there exists a set A that is a finite union of closed cubes such that $m(E\triangle A) < \varepsilon$.

   [*Recall that $E\triangle A$ stands for the symmetric difference, defined by $E\triangle A = (E \setminus A) \cup (A \setminus E)$*]

2. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ with $m(E) > 0$ and $\varepsilon > 0$.

   (a) Prove that E "almost" contains a closed cube in the sense that there exists a closed cube Q such that $m(E \cap Q) \geq (1 - \varepsilon)m(Q)$.

   (b) Prove that the so-called difference set $E-E := \{d : d = x-y \text{ with } x, y \in E\}$ necessarily contains an open ball centered at the origin.
   Hint: It may be useful to observe that $d \in E - E \iff E \cap (E + d) \neq \emptyset$.

3. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* at a point x in $\mathbb{R}^n$ if

$$f(x) \geq \limsup_{y\to x} f(y).$$

   Prove that if f is upper semicontinuous at every point x in $\mathbb{R}^n$, then f is Borel measurable.

4. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}^n$. Prove that

$$\{x \in \mathbb{R}^n : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

   is a measurable set.

5. Recall that the **Cantor set** $\mathcal{C}$ is the set of all $x \in [0, 1]$ that have a ternary expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \neq 1$ for all k. Consider the function

$$f(x) = \sum_{k=1}^{\infty} b_k 2^{-k} \quad \text{where} \quad b_k = a_k/2.$$

   (a) Show that f is well defined and continuous on $\mathcal{C}$, and moreover $f(0) = 0$ as well as $f(1) = 1$.

   (b) Prove that there exists a continuous function that maps a measurable set to a non-measurable set.

6. Let us examine the map f defined in Question 5 even more closely. One readily sees that if $x, y \in \mathcal{C}$ and $x < y$, then $f(x) < f(y)$ unless x and y are the two endpoints of one of the intervals removed from $[0, 1]$ to obtain $\mathcal{C}$. In this case $f(x) = \ell 2^m$ for some integers ℓ and m, and $f(x)$ and $f(y)$ are the two binary expansions of this number. We can therefore extend f to a map $F : [0, 1] \to [0, 1]$ by declaring it to be constant on each interval missing from $\mathcal{C}$. F is called the **Cantor-Lebesgue function**.

   (a) Prove that F is non-decreasing and continuous.

   (b) Let $G(x) = F(x) + x$. Show that G is a bijection from $[0, 1]$ to $[0, 2]$.

   (c) i. Show that $m(G(\mathcal{C})) = 1$.

      ii. By considering rational translates of $\mathcal{N}$ (the non-measurable subset of $[0, 1]$ that we constructed in class), prove that $G(\mathcal{C})$ necessarily contains a (Lebesgue) non-measurable set $\mathcal{N}'$.

      iii. Let $E = G^{-1}(\mathcal{N}')$. Show that E is Lebesgue measurable, but not Borel.

   (d) Give an example of a measurable function φ such that $\varphi \circ G^{-1}$ is not measurable.
   Hint: Let φ be the characteristic function of a set of measure zero whose image under G is not measurable.

Extra Challenge Problems

Not to be handed in with the assignment

1. Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no function f satisfying $f = \chi_{[0,1]}$ almost everywhere which is also continuous on all of $\mathbb{R}$.

2. Question 6d above supplies us with an example that if f and g are Lebesgue measurable, then it does not necessarily follow that $f \circ g$ will be Lebesgue measurable, even if g is assumed to be continuous.

   Prove that if f is Borel measurable, then $f \circ g$ will be Lebesgue or Borel measurable whenever g is.

3. Let f be a measurable function on $[0, 1]$ with $|f(x)| < \infty$ for a.e. x. Prove that there exists a sequence of continuous functions $\{g_n\}$ on $[0, 1]$ such that $g_n \to f$ for a.e. $x \in [0, 1]$.