Math 8100 Exam 2

Wednesday the 27th of November 2018

Answer any <u>THREE</u> of the following four problems

1. Let $f, g \in L^1([0,1])$ and for each $0 \le x \le 1$ define
$$F(x) := \int_0^x f(y)\,dy \quad \text{and} \quad G(x) := \int_0^x g(y)\,dy.$$
Prove that
$$\int_0^1 F(x)g(x)\,dx = F(1)G(1) - \int_0^1 f(x)G(x)\,dx.$$

2. Let $\varphi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ and $\varphi_t(x) := t^{-n}\varphi(t^{-1}x)$. Prove that if $f : \mathbb{R}^n \to \mathbb{C}$ is bounded and uniformly continuous, then $f * \varphi_t$ converges uniformly to f as $t \to 0$.

   Hint: You may assume, with out proof, that for any $\varepsilon > 0$, there exists N such that
$$\int_{|x| \ge N} |\varphi(x)|\,dx < \varepsilon.$$

3. Let $g \in L^\infty([0,1])$.

   (a) Prove that $\lim_{p \to \infty} \|g\|_{L^p([0,1])} = \|g\|_{L^\infty([0,1])}$.

   (b) Let $L^1([0,1])^*$ denote the space of all *continuous linear functional* on $L^1([0,1])$. Prove that the mapping $\Lambda_g : L^1([0,1]) \to \mathbb{C}$ defined by
$$\Lambda_g(f) := \int_0^1 fg$$
   for each $f \in L^1([0,1])$ defines an element of $L^1([0,1])^*$ with norm $\|\Lambda_g\|_{L^1([0,1])^*} = \|g\|_{L^\infty([0,1])}$.

4. Let $\{u_n\}_{n=1}^\infty$ be an orthonormal set in a Hilbert space H.

   (a) Let x be any element of H. Verify that
$$\left\| x - \sum_{n=1}^N \langle x, u_n \rangle u_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, u_n \rangle|^2$$
   for any $N \in \mathbb{N}$ and deduce from this *Bessel's inequality*, namely that
$$\sum_{n=1}^\infty |\langle x, u_n \rangle|^2 \le \|x\|^2.$$

   (b) Let $\{a_n\}$ be any element of $\ell^2(\mathbb{N})$. Prove that there exist $x \in H$ such that $a_n = \langle x, u_n \rangle$ for all $n \in \mathbb{N}$, and moreover that x may be chosen so that
$$\|x\| = \Big(\sum_{n=1}^\infty |a_n|^2 \Big)^{1/2}.$$

   (c) Prove that if $\{u_n\}_{n=1}^\infty$ is *complete*, namely that it has the property that $x = 0$ whenever $\langle x, u_n \rangle = 0$ for all $n \in \mathbb{N}$, then
$$\sum_{n=1}^\infty |\langle x, u_n \rangle|^2 = \|x\|^2 \quad \text{for every } x \in H.$$