Minkowski's Inequality for Integrals

Let $1 \leq p \leq \infty$ and f be measurable on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. If for a.e $y \in \mathbb{R}^{n_2}$

(i) $f^y(x) = f(x,y) \in L^p(\mathbb{R}^{n_1})$

& (ii) $\| f(\cdot, y) \|_p \in L^1(\mathbb{R}^{n_2})$

then for a.e. $x \in \mathbb{R}^{n_1}$

(i) $f_x(y) = f(x,y) \in L^1(\mathbb{R}^{n_2})$

& (ii) $\| f(x, \cdot) \|_1 \in L^p(\mathbb{R}^{n_1})$

Moreover,

$$\left(\int_{\mathbb{R}^{n_1}} \left| \int_{\mathbb{R}^{n_2}} f(x,y)\, dy \right|^p dx \right)^{1/p} \leq \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} |f(x,y)|^p dx \right)^{1/p} dy$$

$$\left[\, \left\| \int f(\cdot, y)\, dy \right\|_p \leq \int \| f(\cdot, y) \|_p \, dy \quad \text{if } p = \infty \, \right]$$

Proof: • $p = 1$: Simply Fubini / Tonelli

• $p = \infty$: Follows immediately from monotonicity.

• $1 < p < \infty$: Let $F(x) = \int_{\mathbb{R}^{n_2}} f(x,y)\, dy$. Note $\|F\|_{L^p(\mathbb{R}^{n_1})} = \text{LHS}$.

Let $g \in L^q(\mathbb{R}^{n_1})$ with $\frac{1}{p} + \frac{1}{q} = 1$. It follows from Fubini / Tonelli that

$$\left| \int F(x) g(x)\, dx \right| \leq \int_{\mathbb{R}^{n_1}} \left(\int_{\mathbb{R}^{n_2}} |f(x,y)| |g(x)|\, dy \right) dx$$

$$= \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} |f(x,y)| |g(x)|\, dx \right) dy$$

Hölder

$$\leq \|g\|_{L^q(\mathbb{R}^{n_1})} \cdot \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} |f(x,y)|^p dx \right)^{1/p} dy .$$

Converse of Hölder

Hence

$$\|F\|_{L^p(\mathbb{R}^{n_1})} = \sup_{\|g\|_q = 1} \left| \int F(x) g(x)\, dx \right| \leq \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} |f(x,y)|^p dx \right)^{1/p} dy . \quad \square$$