Math 4100/6100 **Fall 2015**

Exam 1

1. (10 points) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ with $|x_n| \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$.

   (a) Give the definition of $\lim_{n\to\infty} x_n = x$.

   (b) Prove that $|x| \leq 1$

   (c) Give a direct proof, using the definition given in (a), of the fact that $\lim_{n\to\infty} x_n^2 = x^2$.

2. (10 points) Determine whether the following series converge or diverge, be sure to justify your answer.

   (a) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

   (b) $\sum_{n=1}^{\infty} \frac{(n!)^2 4^n}{(2n)!}$

3. (10 points) Let A and B be non-empty subsets of $\mathbb{R}$ that are bounded above.

   (a) Define what it means for α to be

      i. an upper bound for A,

      ii. a least upper bound for A.

   (b) Let $A + B = \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$.

   (c) Let $A - B = \{a - b : a \in A, b \in B\}$. Must $\sup(A - B) = \sup A - \sup B$? Give either a proof or counterexample.

4. (10 points)

   (a) Let E be a non-empty subset of $\mathbb{R}$ which is bounded above. Prove that $\sup E \in \overline{E}$.

   (b) Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. Using (a), or otherwise, prove that $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$ with the property that

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n.$$

5. (10 points)

   (a) Define finite, countable, and uncountable.

   (b) Let A be an uncountable subset of $[1, \infty)$.

      i. Show that there exist $n \in \mathbb{N}$ such that $A \cap [n, n + 1)$ is uncountable.

      ii. Using (i), or otherwise, prove that A must have a limit point in $[1, \infty)$.

      iii. * Prove that A must in fact contain one of its limit points.