Math 4100/6100 **Fall 2016**

Exam 1

Math 4100: Answer any FOUR of the following SIX questions
Math 6100: Answer any FIVE of the following SIX questions

** All questions are weighted equally*

1. Prove that $F \subseteq \mathbb{R}$ is a closed set if and only if its complement F^c is an open set.

2. (a) Carefully state the *Axiom of Completeness* (the least upper bound axiom).
   (b) Let $\{x_n\}$ be a bounded increasing sequence of real numbers. Use the *Axiom of Completeness* to prove that $\lim_{n\to\infty} x_n$ exists and equals $\sup\{x_n : n \in \mathbb{N}\}$.

3. Let $\{x_n\}$ be a bounded sequence. Prove that if $\beta < \limsup_{n\to\infty} x_n$, then

$$\{n \in \mathbb{N} \, : \, x_n > \beta\} \text{ is infinite}$$

   directly twice, once each using the following equivalent definitions:

   (a) $\limsup_{n\to\infty} x_n := \sup\{x \in \mathbb{R} \, : \, x \text{ is a subsequential limit of } \{x_n\}\}$
   (b) $\limsup_{n\to\infty} x_n := \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k$

4. Let K be a non-empty sequentially compact subset of $\mathbb{R}$.
   (a) Prove that K is both closed and bounded.
   (b) Prove that $\sup K$ exists and is contained in K.

5. (a) Define finite, countable, and uncountable.
   (b) Let A be an uncountable subset of $\mathbb{R}$.
      i. Show that there exist $n \in \mathbb{Z}$ such that $A \cap [n, n+1)$ is uncountable.
      ii. Using i., or otherwise, prove that A must have a limit point in $\mathbb{R}$.

6. Let K be a non-empty compact subset of $\mathbb{R}$ and $\{F_1, F_2, F_3, \dots\}$ be a countable collection of closed subsets of K with the property that

$$\bigcap_{n=1}^{N} F_n \neq \emptyset \text{ for all } N \in \mathbb{N}.$$

   Prove that

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$