Math 4100/6100 Assignment 8 & 9
Continuity and Differentiation

Due date: By 5:00 pm on Thursday the 2nd of November 2017

1. (a) Let

$$f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   i. For which values of a is f_a continuous at 0?

   ii. For which values of a is f_a differentiable at 0? In this case is the derivative function continuous?

   iii. For which values of a is f_a twice-differentiable?

   (b) Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Find particular non-negative (and potentially non-integral) values of a for which:

   i. g_a is differentiable on $\mathbb{R}$, but g'_a is unbounded on $[0, 1]$.

   ii. g_a is differentiable on $\mathbb{R}$ with g'_a continuous but not differentiable at 0.

   iii. g_a and g'_a are differentiable on $\mathbb{R}$, but g''_a is not continuous at 0.

2. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

   (a) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

   (b) Show that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [0, 1]$ satisfying $|x - y| = 1/n$ and $f(x_n) = f(y_n)$.

   (c) Show that if $h \in (0, 1/2)$, but not of the form $1/n$ for some $n \in \mathbb{N}$, then there does not necessarily exist $x, y \in [0, 1]$ satisfying $|x - y| = h$ and $f(x) = f(y)$.

3. Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, lets assume that the functions are defined on all of $\mathbb{R}$.

   (a) Function f and g not differentiable at $x_0 = 0$, but where fg is differentiable at $x_0 = 0$.

   (b) A function f not differentiable at $x_0 = 0$ and a function g differentiable at $x_0 = 0$ where fg is differentiable at $x_0 = 0$.

   (c) A function f not differentiable at $x_0 = 0$ and a function g differentiable at $x_0 = 0$ where $f + g$ is differentiable at $x_0 = 0$.

   (d) A function f differentiable at $x_0 = 0$, but not differentiable at any other point.

4. For each $n \in \mathbb{N}$ and $x \in [0, \infty)$, let

$$f_n(x) = \frac{x}{1 + x^n} \qquad \text{and} \qquad g_n(x) = \begin{cases} 1 & \text{if } x \geq 1/n \\ nx & \text{if } 0 \leq x < 1/n \end{cases}.$$

   Answer the following questions for the sequences $\{f_n\}$ and $\{g_n\}$:

   (a) Find the pointwise limit on $[0, \infty)$.

   (b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.

   (c) Choose a small set over which the convergence is uniform and prove that this is the case.

5. Let

$$f_n(x) = \frac{nx + x^2}{2n} \qquad \text{and} \qquad g_n(x) = \frac{nx^2 + 1}{2n + x}$$

for each $x \in \mathbb{R}$ and $f(x) := \lim_{n\to\infty} f_n(x)$ and $g(x) := \lim_{n\to\infty} g_n(x)$. Show that f and g are both differentiable on $\mathbb{R}$ in two ways: (i) by computing f and g, and (ii) using theorems on uniform convergence.

6. (a) Show that $f(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}$ is continuous on all of $\mathbb{R}$.

   (b) Show that $g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is continuous on $[-1, 1]$.

   (c) Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}.$$

      i. Show that $h(x)$ is a continuous function on $\mathbb{R}$.
      ii. Is h differentiable? If so, is the derivative function h' continuous?

Math 6100/Bonus Problems

7. If $f : [a, b] \to \mathbb{R}$ is one-to-one, then there exists an inverse function f^{-1} defined on the range of f by

$$f^{-1}(y) = x$$

   where $y = f(x)$.

   (a) Prove that if f is continuous on $[a, b]$, then f^{-1} is continuous on its domain.

   (b) Prove that if f is differentiable on $[a, b]$ with $f'(x) \neq 0$ for all $x \in [a, b]$, then f^{-1} is differentiable on its domain with

$$(f^{-1})'(y) = (f'(x))^{-1}$$

   where $y = f(x)$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded.

   (a) We define the *oscillation of f at x* by

$$\operatorname{Osc}(f, x) := \lim_{\delta\to 0^+} \sup_{y,z\in B_\delta(x)} |f(y) - f(z)|.$$

   Briefly explain why this is a well defined notion.

   (b) Prove that f is continuous at x if and only if $\operatorname{Osc}(f, x) = 0$.

   (c) Prove that for every $\varepsilon > 0$ the set $A_\varepsilon = \{x \in \mathbb{R} : \operatorname{Osc}(f, x) \geq \varepsilon\}$ is closed and deduce from this that the set of all points at which f is discontinuous is an F_σ set[1].

Challenge Problem

9. Given an arbitrary F_σ set V, can you produce a function whose discontinuities lie precisely in V?

   Hint: First this for an arbitrary closed set.

[1] Recall that an F_σ set is a countable union of closed sets.