Math 4100/6100 Assignment 6

Continuity and some more Basic Topology of $\mathbb{R}$

Due date: 12:00 pm on Friday the 13th of October 2017

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$\lim_{h\to 0}\big(f(x+h) - f(x-h)\big) = 0$$
for every $x \in \mathbb{R}$. Does this imply that f is continuous?

2. (a) Define *Dirichlet's function* $g : \mathbb{R} \to \mathbb{R}$, by
$$g(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$
Prove that g is discontinuous at $x \in \mathbb{R}$.

   (b) Define a *modified Dirichlet's function* $h : \mathbb{R} \to \mathbb{R}$, by
$$h(x) := \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$
Prove that h is continuous at $x = 0$, but discontinuous at all $x \neq 0$.

   (c) Define *Thomae's function* $t : \mathbb{R} \to \mathbb{R}$, by
$$t(x) := \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{1}{n} & \text{if } x = \dfrac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$
Prove that t is continuous at every $x \notin \mathbb{Q}$, but has a <u>simple</u> discontinuity at every $x \in \mathbb{Q}$.

3. Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that f is defined and continuous on all of $\mathbb{R}$.

   (a) If $f(x) \geq 0$ for all $x < 1$, then $f(1) \geq 0$ as well.

   (b) If $f(r) = 0$ for all $r \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

   (c) If $f(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $f(x)$ is in fact strictly positive for uncountably many points.

4. A set $A \subseteq \mathbb{R}$ is called nowhere-dense if $\overline{A}$ contains no non-empty open intervals.

   (a) Show that a set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

   (b) Decide whether teh following sets are dense in $\mathbb{R}$, nowhere-dense in $\mathbb{R}$, or somewhere in between:

      i. $\mathbb{Q} \cap [0, 1]$

      ii. $\{1/n : n \in \mathbb{N}\}$

      iii. the irrationals $\mathbb{R} \setminus \mathbb{Q}$

      iv. the Cantor set

5. A set $A \subseteq \mathbb{R}$ is called an F_σ set if it can be written as the countable union of closed sets. A set $B \subseteq \mathbb{R}$ is called a G_δ set if it can be written as the countable intersection of open sets $\mathbb{R}$.

   (a) Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

   (b) i. Show that a closed interval $[a, b]$ is a G_δ set.

   ii. Show that a half-open interval $[a, b)$ is both a G_δ set and an F_σ set.

   iii. Show that $\mathbb{Q}$ is an F_σ set and the irrationals $\mathbb{R} \setminus \mathbb{Q}$ is a G_δ set.

   (c) i. Show that every closed set is a G_δ set and every open set is an F_σ set.

   ii. Give an example of an F_σ set which is not a G_δ set.
   Hint: Use the fact that $\mathbb{R}$ cannot be written as a countable union of nowhere-dense sets. Can you recall the proof of this fact?

   iii. Give an example of a set which is neither an F_σ nor a G_δ set.

Math 6100/Bonus Problems

1. Let $C([0, 1])$ denote the collection of all real-valued continuous functions with domain $[0, 1]$.

   (a) Show that $d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$ defines a metric on $C([0, 1])$ and that with the "uniform" metric $C([0, 1])$ is in fact a *complete* metric space.

   (b) Prove that the unit ball $\{f \in C([0, 1]) : d_\infty(f, 0) \leq 1\}$ is closed and bounded, but *not* compact.

   (c) Show that $C([0, 1])$ with the metric d_∞ is not *totally bounded.*
   A set is totally bounded if, for every $\varepsilon > 0$, it can be covered by finitely many balls of radius ε.