Exam 2 Material

Continuity, Differentiation, and Sequences & Series of Functions

Examinable Material

You are responsible for all material discussed in lecture since Exam 1 and the following sections from Rudin:

4.1-4.9: Limits and Continuity (including Sequential Characterizations that where presented in class)
** 4.10-4.11 were not covered and are non-examinable)*
4.13-4.24: The Three C's: Continuity, Compactness, and Connectedness (Theorem 2.47 is non-examinable)
4.25-4.31: Discontinuities and Monotone Functions
** 4.32-4.34 were not covered and are non-examinable*

5.1-5.11: Derivatives, the Mean Value Theorem, and it's consequences
5.12: Derivatives have the Intermediate Value Property
** 5.13 was not covered and is non-examinable*
5.15: Taylor's Theorem
** 5.16-5.19 were not covered and are non-examinable*

7.1-7.6: Examples of Sequences & Series of Functions
7.7-7.10: Uniform Convergence
7.11-7.12: Uniform Convergence and Continuity
** 7.13 was not covered and is non-examinable*
7.14-7.14: $C([0,1])$ is a complete metric space
Plus the fact, discussed in class, that the "closed unit ball" in $C([0,1])$ is not compact
** 7.16 was not covered and is non-examinable*
7.17: Uniform Convergence and Differentiation (statement only for this exam)
7.18: Example of a nowhere differentiable function (statement only for this exam)
** 7.19-7.25 were not covered and are non-examinable*
7.26: Weierstrass Approximation Theorem (statement only for this exam)
** 7.27-7.33 were not covered and are non-examinable*

8.1-8.2: Power Series including Theorem 3.39 (only the statement of Theorem 8.2 on this exam)

Sample Exam 2 Questions

1. (a) Let $X \subseteq \mathbb{R}$, $x_0 \in X$, and $f : X \to \mathbb{R}$.
Carefully state the ε-δ definition of what it means for f to be *continuous* at x_0.

   (b) Use the definition from part (a) to prove that
$$f(x) = \frac{x^2 + 5x - 2}{x + 1}$$
is continuous at 2.

   (c) Prove that if $\lim_{n\to\infty} f(x_n) = f(x_0)$ for all sequences with $\lim_{n\to\infty} x_n = x_0$, then f is continuous at x_0.

   (d) Let
$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

   i. Carefully argue that g is discontinuous at every point of $\mathbb{R}$.
   ii. Let $h(x) = xg(x)$ for every $x \in \mathbb{R}$. Prove that h is continuous at 0, but is not differentiable at 0.
   iii. Give an example, no proof is required, of a function that is defined on all of $\mathbb{R}$ but is only differentiable at 0.

2. Suppose $f'(x_0)$, $g'(x_0)$ exist, $g'(x_0) \neq 0$, and $f(x_0) = g(x_0) = 0$. Prove that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

3. (a) Carefully state the *Mean Value Theorem* and use it to prove the following:
   i. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in \mathbb{R}$, then f must be constant on $\mathbb{R}$.
   ii. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) \geq 0$ for all $x \in (0, \infty)$, then f is increasing on $(0, \infty)$.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ has the property that

$$|f(x) - f(y)| \leq |x - y|^2$$

   for all $x, y \in \mathbb{R}$. Prove that f is constant on $\mathbb{R}$.

   (c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, and f' is increasing on $(0, \infty)$. Prove that the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \frac{f(x)}{x}$$

   is increasing.

4. (a) Give a definition of what it means for a function $f : X \to \mathbb{R}$ to be *uniformly continous* on X.

   (b) Give an example of a function that is continuous on a set X but fails to be uniformly continuous on X (no proofs required).

   (c) Prove that a function that is continuous on a compact set K is uniformly continuous on K.

5. Let f be a differentiable function on $[a, b]$. We say that f is *uniformly differentiable* on $[a, b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon$$

whenever $|x - y| < \delta$ with $x, y \in [a, b]$.

   (a) Prove that f is uniformly differentiable on $[a, b]$ <u>if and only if</u> f' is continuous on $[a, b]$.

   (b) Give an example of a function that is differentiable on $[a, b]$ but fails to be uniformly differentiable on $[a, b]$ (no proofs required).

6. (a) Let $\{r_n\}$ be any enumeration of all the rationals in $[0, 1]$ and define $f : [0, 1] \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = r_n \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases} .$$

   Prove that $\lim_{x \to c} f(x) = 0$ for every $c \in [0, 1]$ and conclude that set of all points at which f is discontinuous is precisely $[0, 1] \cap \mathbb{Q}$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and
   i. Recall that the *oscillation of* f *at* x is define to be

$$\text{Osc}(f, x) := \lim_{\delta \to 0^+} \sup_{y, z \in B_\delta(x)} |f(y) - f(z)|.$$

   Briefly explain why this is a well defined notion and prove that

$$f \text{ is continuous at } x \iff \text{Osc}(f, x) = 0.$$

   ii. Prove that for every $\varepsilon > 0$ the set $A_\varepsilon = \{x \in \mathbb{R} : \text{Osc}(f, x) \geq \varepsilon\}$ is closed and deduce from this that the set of all points at which f is discontinuous is an F_σ set.

7. (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set X.

   (b) Consider the sequence of functions
$$f_n(x) = \frac{x}{1+nx^2}.$$
   Find the points on $\mathbb{R}$ where each $f_n(x)$ attains it maximum and minimum value. Use this to prove that $\{f_n\}$ converges uniformly on $\mathbb{R}$.

   (c) Consider the sequence of functions
$$g_n(x) = \frac{x+n}{n}$$
      i. Find the pointwise limit of $\{g_n\}$ on $\mathbb{R}$.
      ii. Show that $\{g_n\}$ does not converge uniformly on $\mathbb{R}$.
      iii. Show that $\{g_n\}$ does converge uniformly on $[-M, M]$ for any $M > 0$.

   (d) Prove that $h_n(x) = x^n(1-x)$ converges uniformly to 0 on $[0, 1]$.

8. (a) Define what it means to say that a series $\sum_{n=0}^{\infty} f_n$ *converge uniformly* on X to a sum function s.

   (b) State and prove the Weierstrass M-test for the uniform convergence of $\sum_{n=0}^{\infty} f_n$.

   (c) Prove that if $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on a set X, then the sequence of functions $\{g_n\}$ must converge uniformly to 0 on X.

9. (a) Show that $\sum\limits_{n=1}^{\infty} \dfrac{x}{1+x^n}$ diverges for all $x \in (0, 1]$.

   (b) Let
$$f(x) = \sum_{n=0}^{\infty} \frac{x}{1+x^n}.$$
      i. Show that the series defining f does not converge uniformly on $(1, \infty)$, but none the less still defines a continuous function on $(1, \infty)$
      ii. Is f differentiable? If so, is the derivative function f' continuous?

10. Let
$$g(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2x}.$$
   (a) Show that the series defining g does not converge uniformly on $(0, \infty)$, but none the less still defines a continuous function on $(0, \infty)$.

   (b) Is g differentiable? If so, is the derivative function g' continuous?

11. Let $h_n(x) = \dfrac{x}{(1+x)^{n+1}}$.

   (a) Prove that h_n converges uniformly to 0 on $[0, \infty)$.

   (b) i. Verify that
$$\sum_{n=0}^{\infty} h_n(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \end{cases}$$
      ii. Does $\sum_{n=0}^{\infty} h_n$ converge uniformly on $[0, \infty)$?

   (c) Prove that $\sum_{n=0}^{\infty} h_n$ converges uniformly on $[a, \infty)$ for any $a > 0$.

Hint: Recall that the Binomial Theorem implies $(1+x)^{n+1} \geq \dfrac{n(n+1)}{2}x^2$ *for all* $x \geq 0$.