Exam 1 Material

Properties of Completeness, Cardinality of Sets, Basic Topology (Metric Spaces), and Numerical Sequences and Series

<u>Examinable Material</u> (References are Sections from Rudin)

1.1-1.23: Axioms of $\mathbb{R}$ (we focused specifically on 1.7-1.11, 1.19-1.21)
** 1.24-1.35: Complex Numbers (non-examinable)*
1.36-1.38: Euclidean Spaces (including the Cauchy-Schwarz inequality)

2.1-2.14: Cardinality (including proofs given in class that $\mathbb{R}$ is uncountable)
2.15-2.30: Metric Spaces
** 2.29-2.30 were not covered are are non-examinable*
2.31-2.42: Compact Sets (Class notes differs from text)
** 2.33, 2.36, 2.39, 2.40 and 2.45-2.47 were not covered are are non-examinable*
** 2.43-2.44: Perfect sets (non-examinable, for this Exam)*

3.1-3.14: Numerical Sequences
3.15-3.20: Limit superior, limit inferior, and some special sequences
3.21-3.46: Numerical Series (including Dirichlet and Abel tests) - See class handout
** 3.38-3.40: Power Series (non-examinable)*
** 3.47-3.55: Addition and Multiplication of Series and Rearrangements (non-examinable)*

Exam 1 Practice Questions

1. (a) Give the definition of the infimum of a bounded set of real numbers.
   (b) Let $\{x_n\}$ be a bounded sequence. Give the definition of $\liminf x_n$ and prove that it always exists in $\mathbb{R}$.
   (c) Prove that if $\beta < \liminf x_n$, then there exists a $N \in \mathbb{N}$ such that $x_n > \beta$ for all $n \geq N$.
2. (a) Give a definition of what it means for a set to be countable (recall that a countable set must be infinite).
   (b) Prove that the natural numbers can be written as a countable union of disjoint countable sets.
   (c) Using the observation in (b) (or otherwise) to prove that a countable union of countable sets is always countable.
3. Give examples of the following. No proofs are required.
   (a) a divergent series whose sequence of partial sums is bounded
   (b) an infinite subset of $\mathbb{R}$ which has no limit points,
   (c) a proper subset of $\mathbb{R}$ which has uncountably many limit points
   (d) a bounded set of real numbers with countably many limit points
   (e) an open cover of $(1, 2]$ which does not admit a finite subcover
4. (a) State and prove the Bolzano-Weierstrass theorem.
   (b) Prove that every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence, using the fact that we know this to be true in the case $k = 1$.
5. Prove that every non-empty compact subset of $\mathbb{R}$ has a largest member.