Math 4100/6100 **Fall 2017**

Exam 1

Math 4100: Answer any THREE of the following FIVE questions
Math 6100: Answer any FOUR of the following FIVE questions

** All questions are weighted equally.*

In Questions 1-4 below the real numbers $\mathbb{R}$ are equipped with their usual Euclidean metric.

1. (a) Prove that if $\{x_n\}$ is a sequence of real numbers with the property that the set
$$\{n \in \mathbb{N} : x_n \in B_\varepsilon(x)\}$$
is infinite for all $\varepsilon > 0$, then $\{x_n\}$ must contain a subsequence which converges to x.

   (b) Let $\sum_{n=1}^\infty a_n$ be a convergent series of real numbers and $\{b_n\}$ be a monotone and bounded sequence of real numbers. Prove that $\sum_{n=1}^\infty a_n b_n$ converges.
Hint: First show that if $s_n = a_1 + \cdots + a_n$ for all $n \geq 1$, then
$$\sum_{k=1}^n a_k b_k = s_n b_{n+1} + \sum_{k=1}^n s_k(b_k - b_{k+1}).$$

2. Let $\{x_n\}$ be a sequence with the property that $x_n > 0$ for all $n \in \mathbb{N}$.

   (a) Give a definition of $\liminf_{n\to\infty} x_n$ and carefully argue why it is well-defined and either takes values in $[0, \infty)$ or equals ∞.
Be sure to indicate under what circumstances $\liminf_{n\to\infty} x_n = \infty$.

   (b) i. Prove that if $\alpha := \liminf_{n\to\infty} \dfrac{x_{n+1}}{x_n}$, then $\liminf_{n\to\infty} x_n^{1/n} \geq \alpha$.
   ii. Give an example where one has strict inequality above, no proof is required.

3. (a) Prove that any collection of open intervals in $\mathbb{R}$ with the property that no three contain a common element is at most countable.

   (b) Let $A \subseteq \mathbb{R}$ be uncountable.
   i. Prove that the isolated points of A form an at most countable set.
   ii. Conclude that $A \cap A'$, the set of limit points of A contained in A, is uncountable and further prove that for any $\varepsilon > 0$, there in fact exists $x \in \mathbb{R}$ such that $A \cap A' \cap B_\varepsilon(x)$ is uncountable.

4. Let K be a non-empty sequentially compact subset of $\mathbb{R}$.

   (a) Prove that K is both closed and bounded.
   (b) Prove that $\sup K$ exists and is contained in K.

5. Let (X, d) be a metric space.

   (a) Prove that if $G \subseteq X$ is an open set, then its complement G^c is a closed set.
   (b) Let y, z be fixed points in X. Prove that the set
$$E = \{x \in X : d(x, y) \geq d(x, z)\}$$
is closed.