Math 3100
Sample Exam 3 – Version 2

No calculators. Show your work. Give full explanations. Good luck!

1. (7 points)
   (a) Carefully state the *Intermediate Value Theorem.*
   (b) Let f be a continuous function on the closed interval $[0,1]$ with range also contained in $[0,1]$. Prove that f must have a fixed point; that is, show that $f(x) = x$ for at least one value of $x \in [0,1]$.

2. (10 points) Let $f(x) = \begin{cases} x^4 \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$
   (a) Show that f is differentiable at 0 and compute $f'(x)$ for all $x \in \mathbb{R}$.
   (b) Is f' continuous at 0? Give your reasoning.
   (c) Is f' differentiable at 0? Give your reasoning.

3. (8 points)
   (a) Find the 4th order Maclaurin polynomial for $f(x) = \dfrac{\cos(x^2)}{1+x}$.
   (b) Use part (a) to find the value of $f^{(4)}(0)$ without differentiating.

4. (10 points)
   (a) Carefully state the *Lagrangian Remainder Estimate* for Maclaurin series.
   (b) Use the *Lagrangian Remainder Estimate* to determine the following:
      i. An estimate for the accuracy of approximating $\sin x$ by $x - x^3/6$ when $|x| \leq 1/2$.
      ii. Values of x for which the accuracy of approximating $\sin x$ by $x - x^3/6$ is less than 10^{-3}.

   Note that you are not permitted to use the Alternating Series Remainder Estimate above.
   (c) Obtain, by any means, an estimate for the accuracy of approximating
$$\int_0^1 \frac{\sin x}{x}\, dx \quad \text{by} \quad 1 - \frac{1}{18}.$$

5. (15 points)
   (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.
   (b) Consider the sequence of functions
$$f_n(x) = \frac{x}{1+x^n}.$$
      i. Find the pointwise limit of $\{f_n\}$ on $[0,\infty)$.
      ii. Explain how we know that the convergence cannot be uniform on $[0,\infty)$.

   (c) i. Show that $\sum_{n=1}^{\infty} \dfrac{x}{1+x^n}$ diverges for all $x \in (0,1]$, but converges if $x > 1$.
      ii. Let $f(x) = \sum_{n=1}^{\infty} \dfrac{x}{1+x^n}$ on $(1,\infty)$.
         A. Prove that the series defining f does not converge uniformly on $(1,\infty)$.
         B. Prove that f is a continuous function on $(1,\infty)$.

   Hint: Show that the series defining f converges uniformly on $[a,\infty)$ for any $a > 1$.