Math 3100
Sample Exam 2 – Version 0

No calculators. Show your work. Give full explanations. Good luck!

1. (15 points)

   (a) Carefully state the definition of what it means to say that $\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) i. Prove that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   ii. Is it true that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent? Give either a proof or counterexample.

   (c) Let $b_n \geq 0$ for all $n \in \mathbb{N}$.

   i. Prove that if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n^2$ also converges and that in fact

   $$\sum_{n=1}^{\infty} b_n^2 \leq \left(\sum_{n=1}^{\infty} b_n\right)^2.$$

   ii. Is it true that if $\sum_{n=1}^{\infty} b_n^2$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges? Give either a proof or counterexample.

2. (20 points)

   (a) Determine if the following series are absolutely convergent, conditionally convergent, or divergent. Justify your answers.

   $$\text{(i)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+1} \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1} \qquad \text{(iii)} \quad \sum_{n=1}^{\infty} \frac{(\log n)^3}{n^2}$$

   (b) Find all $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$ converges.

   (c) Find a sequence $\{a_n\}$ so that $\sum_{n=2}^{\infty} a_n x^n = \frac{x^2}{2+x}$ for all $|x| < 2$.

3. (15 points)

   (a) i. Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$. Carefully state the ε-δ definition of what it means for f to be *continuous* at a point $x_0 \in X$.

   ii. Use this ε-δ definition to prove that $f(x) = \frac{3-x}{x^2}$ is continuous at $x_0 = 2$.

   Hint: Use the fact that $\left|\frac{3-x}{x^2} - \frac{1}{4}\right| = \frac{|x+6|}{4x^2}|x-2|$

   (b) i. Let $g : \mathbb{R} \to \mathbb{R}$. Carefully state the *sequential characterization* of what it means for g to be *continuous* at a point $x_0 \in \mathbb{R}$.

   ii. Prove, using the sequential characterization or otherwise, that the function

   $$g(x) = \begin{cases} x & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

   is not continuous at $x_0 = 1$.