# Math 3100

## Sample Exam 1 – Version 2

*No calculators. Show your work. Give full explanations. Good luck!*

1. (8 points) Give counterexamples to the following **false** statements, no proofs are required.

   *Note that in each instance the converse statement is in fact true.*

   (a) If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent.

   (b) If $\{x_n\}$ is convergent, then $\{x_n\}$ is both bounded and monotone.

   (c) If $\{x_n\}$ contains a convergent subsequence, then $\{x_n\}$ is bounded.

   (d) If $A$ contains its supremum, then $A$ has finitely many elements.

2. (4 points) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. Prove that if $\lim_{n\to\infty} x_n = x$ and $|x_n - y_n| \le \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} y_n = x$.

3. (14 points)

   (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of $\lim_{n\to\infty} x_n = x$.

   (b) Use your definition to prove that $\lim_{n\to\infty} \frac{3n+4}{n+1} = 3$.

   (c) Assume that $\lim_{n\to\infty} x_n = x > 0$. Using <u>only</u> the definition of convergence prove the following two statements;

      i. there exists a number $N$ such that if $n > N$, then $x_n > \frac{1}{2}x$.

      ii. $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$

4. (14 points)

   (a) Carefully state the *Axiom of Completeness* (the least upper bound axiom).

   (b) Let $\{x_n\}$ be a bounded increasing sequence of real numbers. Use the *Axiom of Completeness* to prove that $\lim_{n\to\infty} x_n$ exists and equals $\sup\{x_n : n \in \mathbb{N}\}$.

   (c) Prove that if $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges and find the value of its limit.

5. (10 points) Let $\{x_n\}$ be a sequence of real numbers that satisfy the property that $|x_n| \le 1$ for all $n \in \mathbb{N}$.

   (a) Prove that <u>if</u> $\lim_{n\to\infty} x_n$ exists and equals $x$, then $x \le 1$.

   (b) Carefully explain why $\limsup_{n\to\infty} x_n$ exists and why $\limsup_{n\to\infty} x_n \le 1$.