# Math 3100 Assignment 2

## Sequences: Boundedness, Monotonicity, and Convergence

*Due at the 5:00 pm on Friday the 25th of January 2019*

1. Which of the sequences below are increasing, strictly increasing, decreasing, strictly decreasing, or none of the above? Justify your answers. Which are bounded above, or bounded below; which are bounded? Give an upper bound and/or lower bound when applicable.

   (a) $a_n = n^2 - n$

   (b) $b_n = \dfrac{1}{n+1}$

   (c) $c_n = \dfrac{(-1)^n}{n^3}$

   (d) $x_{n+1} = x_n + \dfrac{1}{(n+1)^2}$, for $n \in \mathbb{N}$ and $x_1 = 1$

   (e) $y_n = 17$ for all $n \in \mathbb{N}$

   *Challenge: Can you show that the sequence defined by* $x_{n+1} = x_n + \dfrac{1}{n+1}$, *for* $n \in \mathbb{N}$ *and* $x_1 = 1$ *is strictly increasing and* <u>*not*</u> *bounded above.*

2. (a) Let $\{a_n\}$ be a sequence given recursively by $a_{n+1} = \dfrac{3a_n + 2}{a_n + 2}$ with $a_1 = 3$.
   Prove that $\{a_n\}$ is decreasing and satisfies $a_n \geq 2$ for all $n \in \mathbb{N}$.

   (b) Let $\{a_n\}$ be a sequence given recursively by $a_{n+1} = \dfrac{4a_n + 3}{a_n + 2}$ with $a_1 = 2$.
   Prove that $\{a_n\}$ is increasing and satisfies $2 \leq a_n \leq 3$ for all $n \in \mathbb{N}$.

   (c) Let $\{b_n\}$ be a sequence given recursively by $b_{n+1} = \dfrac{b_n}{2} + \dfrac{1}{b_n}$ with $b_1 = 2$.
   Use induction to prove that $\{b_n\}$ satisfies both $b_n > 0$ and $b_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$. Use this to establish that $\{b_n\}$ is a decreasing sequence.

3. (a) Let $q \neq 0$ be rational and $x$ be irrational. Prove that $q + x$ and $qx$ are both irrational.

   (b) Give examples of the following:

      i. A sequence $\{x_n\}$ of irrational numbers whose limit is a rational number.

      ii. A sequence $\{q_n\}$ of rational numbers whose limit is an irrational number.

4. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^{1/5}} = 0$ (b) $\displaystyle\lim_{n\to\infty} \frac{2n+1}{3n-2} = \frac{2}{3}$ (c) $\displaystyle\lim_{n\to\infty} \frac{1}{5n^2+2} = 0$

5. Determine the value of the following limits, and then prove your claims using the definition of convergence of a sequence.

   (a) $\displaystyle\lim_{n\to\infty} \frac{2n}{n^3+1}$ (b) $\displaystyle\lim_{n\to\infty} \frac{5n+2}{3n-1}$ (c) $\displaystyle\lim_{n\to\infty} \frac{\cos n}{n^{1/3}}$