Math 3100 Assignment 10

Uniform Convergence

Homework due date: 5:00 pm on Monday the 22nd of April 2019

1. Consider the sequence of functions
$$f_n(x) = \frac{x+n}{n}$$
   (a) Find the pointwise limit of $\{f_n\}$ on $\mathbb{R}$.
   (b) Show that $\{f_n\}$ does not converge uniformly on $\mathbb{R}$.
   (c) Show that $\{f_n\}$ does converge uniformly on $[-M, M]$ for any $M > 0$.

2. Consider the sequence of functions
$$g_n(x) = \frac{x}{1+x^n}.$$
   (a) Find the pointwise limit of $\{g_n\}$ on $[0, \infty)$.
   (b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.
   (c) Write down a smaller set over which the convergence is uniform, no proofs required.

3. (a) Consider the sequence of functions
$$F_n(x) = \frac{x}{1+nx^2}.$$
   Find the points on $\mathbb{R}$ where each $F_n(x)$ attains it maximum and minimum value. Use this to prove that $\{F_n\}$ converges uniformly on $\mathbb{R}$.
   (b) Prove that $G_n(x) = x^n(1-x)$ converges uniformly to 0 on $[0, 1]$.

4. (a) Prove that if $\sum_{n=0}^{\infty} h_n(x)$ converges uniformly on a set A, then the sequence of functions $\{h_n\}$ must converge uniformly to 0 on A.
   (b) Let
$$h(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2x}.$$
   i. Prove that the series defining h does not converge uniformly on $(0, \infty)$.
   ii. Prove that h is however a continuous function on $(0, \infty)$.

5. Let $g_n(x) = \dfrac{nx^2}{n^3+x^3}$.
   (a) Prove that g_n converge uniformly to 0 on $[0, M]$ for any $M > 0$, but does <u>not</u> converge uniformly to 0 on $[0, \infty)$.
   (b) i. Prove that $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[0, M]$ for any $M > 0$.
   ii. Does $\sum_{n=1}^{\infty} g_n$ converge uniformly on $[0, \infty)$?
   iii. Does $\sum_{n=1}^{\infty} g_n$ define a continuous function on $[0, \infty)$?