Math 3100 Assignment 1

Preliminaries

Due at the beginning of class on Wednesday the 16th of January 2019

1. (Induction)

   (a) Prove, by induction, that the following identities hold for all $n \in \mathbb{N}$:

   i. $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

   ii. $1^3 + 2^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

   It seems like the two identities above must be closely related to each other.
   Challenge: Can you give a geometric proof of the second identity using only the first?

   (b) Prove, by induction, that the following inequalities hold for all $n \in \mathbb{N}$:

   i. $2n + 1 \leq 3n^2$

   ii. $2n^2 - 1 \leq n^3$

   Hint: The validity of the first inequality should help you establish the second.

2. (Absolute Value and Inequalities)

   (a) i. If $|x| < 2$, what can you say about $|x - 3|$?

   ii. If $|x - 2| < 1$, what can you say about $|x + 3|$?

   iii. If $|x + 1| < 1/2$, what can you say about $|x|^{-1}$?

   (b) Use the triangle inequality to show that

   $$\big||x| - |y|\big| \leq |x - y|$$

   for all $x, y \in \mathbb{R}$. This inequality is often refered to as the *reverse triangle inequailty.*
   Hint: Start by writing $x = (x - y) + y$.

3. (Irrational numbers are dense in the reals)

   (a) Let $q \in \mathbb{Q}$, that is let q be a rational number. Prove that $q + \sqrt{2}$ must be irrational.

   (b) Prove that given any two real numbers $x < y$, there exists an irrational number z such that $x < z < y$.
   Hint: Try to deduce this as a consequence of the denseness of rationals in the real, and part (a), after considering the real numbers $x - \sqrt{2}$ and $y - \sqrt{2}$.