Math 3100 **Spring 2019**

Exam 3
Study Guide and Practice Questions

1. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$\lim_{h\to 0}\big(f(x+h) - f(x-h)\big) = 0.$$
Does this imply that f is continuous at x?

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at x and satisfies
$$\lim_{h\to 0}\frac{f(x+h) - f(x-h)}{2h} = 0.$$
Does this imply that f is differentiable at x?

2. Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$.

   (a) Carefully state the ε-δ definition of $\lim_{x\to c} f(x) = L$.

   (b) Prove that $\lim_{x\to x_0} f(x) = L$ if and only if $\lim_{n\to\infty} f(x_n) = L$ for all sequences $\{x_n\}$ in $\mathbb{R} \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$.

3. Let
$$f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

   (a) For which values of a is f_a continuous at 0?

   (b) For which values of a is f_a differentiable at 0? In this case is the derivative function continuous?

   (c) For which values of a is f_a twice-differentiable?

4. Let
$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$
Find particular non-negative (and potentially non-integer) values of a for which:

   (a) g_a is differentiable on $\mathbb{R}$, but g_a' is unbounded on $[0, 1]$.

   (b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at 0.

   (c) g_a and g_a' are differentiable on $\mathbb{R}$, but g_a'' is not continuous at 0.

5. (a) State and prove *Rolle's Theorem.*

   (b) Carefully state the *Mean Value Theorem* and use it to prove the following:

      i. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in \mathbb{R}$, then f must be constant on $\mathbb{R}$.

      ii. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) \ge 0$ for all $x \in (0, \infty)$, then f is increasing on $(0, \infty)$.

   (c) Suppose $f : \mathbb{R} \to \mathbb{R}$ has the property that
$$|f(x) - f(y)| \le |x - y|^2$$
for all $x, y \in \mathbb{R}$. Prove that f is constant on $\mathbb{R}$.

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, and f' is increasing on $(0, \infty)$. Prove that the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \frac{f(x)}{x}$$

is increasing.

6. (a) State the *Generalized Mean Value Theorem.* Do you remember it's proof?

   (b) Let f and g be continuous functions on some interval I containing c and differentiable on the same interval except possibly at c itself. Prove that if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

7. (a) Find the value of $f^{(17)}(0)$ if $f(x) = \dfrac{4x}{2 - x}$.

   (b) Give an example of an infinitely differentiable function that is <u>not</u> equal to its Taylor series.

   (c) i. Prove that if $h : [0, \infty) \to \mathbb{R}$ is twice differentiable on $[0, x]$, then

$$h(x) = h(0) + h'(0)x + \frac{h''(c)}{2}x^2$$

   for some $c \in (0, x)$. *Hint: Apply the "Generalized MVT" to* $h(x) - h(0) - h'(0)x$ *and* x^2.

   ii. How well does $1 + x/2$ approximate $\sqrt{1 + x}$ on $[0, 1/10]$?

8. Prove that $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of converegnce.

9. (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.

   (b) Prove that $f_n(x) = \dfrac{nx + \sin(nx^2)}{n}$ converges uniformly to x on $\mathbb{R}$.

   (c) Prove that $F_n(x) = x^n(1 - x)$ converges uniformly to 0 on $[0, 1]$.

10. (a) Carefully state the *Weiertrass M-Test.*

    (b) Show that

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

    is continuous on all of $\mathbb{R}$.

    (c) Let

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{1 + x^{2n}}.$$

    Find the values of x where the series defining g converges and show that we get a continuous function on this set.

11. ** Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$.

    Prove that if $f'(a) < 0 < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.