Math 3100 **Spring 2018**

Sample Exam 3 – Version 2

No calculators. Show your work. Give full explanations. Good luck!

1. (7 points)

   (a) Carefully state the *Intermediate Value Theorem.*

   (b) Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show that $f(x) = x$ for at least one value of $x \in [0, 1]$.

2. (15 points)

   (a) Carefully state the *Mean Value Theorem* and use it to prove the following:

      i. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 0$ for all $x \in \mathbb{R}$, then f must be constant on $\mathbb{R}$.

      ii. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) \geq 0$ for all $x \in (0, \infty)$, then f is increasing on $(0, \infty)$.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ has the property that
   $$|f(x) - f(y)| \leq |x - y|^2$$
   for all $x, y \in \mathbb{R}$. Prove that f is constant on $\mathbb{R}$.

   (c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, and f' is increasing on $(0, \infty)$. Prove that the function $g : (0, \infty) \to \mathbb{R}$ defined by
   $$g(x) = \frac{f(x)}{x}$$
   is increasing.

3. (10 points) Let $f(x) = \begin{cases} x^4 \sin(x^{-2}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$

   (a) Show that f is differentiable at 0 and compute $f'(x)$ for all $x \in \mathbb{R}$.

   (b) Is f' continuous at 0? Give your reasoning.

   (c) Is f' differentiable at 0? Give your reasoning.

4. (8 points)

   (a) Find the 4th order Maclaurin polynomial for $f(x) = \dfrac{\cos(x^2)}{1 + x}$.

   (b) Use part (a) to find the value of $f^{(4)}(0)$ without differentiating.

5. (10 points)

   (a) Carefully state the *Lagrangian Remainder Estimate* for Maclaurin series.

   (b) Use the *Lagrangian Remainder Estimate* to determine the following:

      i. An estimate for the accuracy of approximating $\sin x$ by $x - x^3/6$ when $|x| \leq 1/2$.

      ii. Values of x for which the accuracy of approximating $\sin x$ by $x - x^3/6$ is less than 10^{-3}.

   Use are not permitted to use the Alternating Series Remainder Estimate above.

   (c) Obtain, by any means, an estimate for the accuracy of approximating
   $$\int_0^1 \frac{\sin x}{x}\, dx \quad \text{by} \quad 1 - \frac{1}{18}.$$