Math 3100 Assignment 4
Subsequences and Completeness

Due at 12:00 pm on Friday the 9th of February 2018

1. Suppose $\{a_n\}$ is a sequence of real numbers and $b_n = \dfrac{a_1 + \cdots + a_n}{n}$.

   Prove that if a_n converges to 0, then b_n converges to 0 as well. Is the converse true?

2. (a) Prove that if $\{a_n\}$ is increasing, then every subsequence of $\{a_n\}$ is also increasing.

   (b) Let $\{x_n\}$ be a sequence of real numbers.
   Prove that $\{x_n\}$ contains a subsequence converging to x <u>if and only if</u> for all $\varepsilon > 0$ there exist infinitely many terms from $\{x_n\}$ that satisfy $|x_n - x| < \varepsilon$.

3. Let $A, B \subseteq \mathbb{R}$ which are non-empty, bounded above.

   (a) Show that if $A \subseteq B$, then $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.

   (b) Show that if $\sup A < \sup B$, then there must exist $b \in B$ that is an upper bound for A.

   (c) Prove that if $\sup(A) \notin A$, then there exists a sequence $\{a_n\}$ of points in A such that

   $$\lim_{n\to\infty} a_n = \sup(A).$$

4. (a) Let $a_1 = \sqrt{2}$, and define

   $$a_{n+1} = \sqrt{2 + a_n}$$

   for all $n \geq 1$. Prove that $\lim\limits_{n\to\infty} a_n$ exists and equals 2.

   (b) Let $a_1 = \sqrt{2}$, and define

   $$a_{n+1} = \sqrt{2a_n}$$

   for all $n \geq 1$. Prove that $\lim\limits_{n\to\infty} a_n$ exists and find its limit.

   Hint: For both parts try to apply the Monotone Convergence Theorem

5. Let $\{x_n\}$ be a bounded sequence of real numbers.

   (a) Prove that the "supremum of the tails of $\{x_n\}$" defined by $y_n := \sup\{x_n, x_{n+1}, \dots\}$ is a decreasing sequence that is bounded below. Conclude, using the Monotone Convergence Theorem, that $\{y_n\}$ is convergent.
   The value of $\lim\limits_{n\to\infty} y_n$ is called the *limit superior* of $\{x_n\}$ is usually denoted by $\limsup\limits_{n\to\infty} x_n$.

   (b) The *limit inferior* of $\{x_n\}$, denoted by $\liminf\limits_{n\to\infty} x_n$, is similarly defined to be $\lim\limits_{n\to\infty} z_n$ where $z_n := \inf\{x_n, x_{n+1}, \dots\}$ is the corresponding "infimum of the tails of $\{x_n\}$".
   Briefly explain why the *limit inferior* always exists for any bounded sequence.

   (c) Prove that $\liminf\limits_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$ for every bounded sequence $\{x_n\}$ and give an example for which the inequality is strict.

   (d) Prove that $\lim\limits_{n\to\infty} x_n$ exists <u>if and only if</u> $\liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n$.
   In this case all three share the same value.

   (e) * Let $x := \limsup\limits_{n\to\infty} x_n$. Prove that there exists a subsequence of $\{x_n\}$ that converges x.
   Hint: Try to show that for all $\varepsilon > 0$ there are infinitely many terms from $\{x_n\}$ that satisfy $x - \varepsilon < x_n < x + \varepsilon$ and apply Question 2(b).