Math 3100 Assignment 2

Sequences: Boundedness, Monotonicity, and Convergence

Due at the beginning of class on Friday the 26th of January 2018

1. Which of the sequences below are increasing, strictly increasing, decreasing, strictly decreasing, or none of the above? Justify your answers. Which are bounded above, or bounded below; which are bounded? Give an upper bound and/or lower bound when applicable.

   (a) $a_n = n^2 - n$

   (b) $b_n = \dfrac{1}{n+1}$

   (c) $c_n = \dfrac{(-1)^n}{n^3}$

   (d) $x_{n+1} = x_n + \dfrac{1}{(n+1)^2}$, for $n \in \mathbb{N}$ and $x_1 = 1$

   (e) $y_n = 17$ for all $n \in \mathbb{N}$

   Challenge: Can you show that the sequence defined by $x_{n+1} = x_n + \dfrac{1}{n+1}$, *for* $n \in \mathbb{N}$ *and* $x_1 = 1$ *is strictly increasing and* <u>*not*</u> *bounded above.*

2. (a) Let $\{a_n\}$ be a sequence given recursively by $a_{n+1} = \dfrac{3a_n + 2}{a_n + 2}$ with $a_1 = 1$.
   Prove that $\{a_n\}$ is increasing and satisfies $a_n \leq 2$ for all $n \in \mathbb{N}$.
   Hint: Depending on your approach it may help to also verify that $a_n \geq 0$ *for all* $n \in \mathbb{N}$.

   (b) Let $\{b_n\}$ be a sequence given recursively by $b_{n+1} = \dfrac{b_n}{2} + \dfrac{1}{b_n}$ with $b_1 = 2$.
   Use induction to prove that $\{b_n\}$ satisfies both $b_n > 0$ and $b_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$. Use this to establish that $\{b_n\}$ is a decreasing sequence.

3. (a) Let $q \neq 0$ be rational and x be irrational. Prove that $q + x$ and qx are both irrational.

   (b) Give examples of the following:

      i. A sequence $\{x_n\}$ of irrational numbers whose limit is a rational number.

      ii. A sequence $\{q_n\}$ of rational numbers whose limit is an irrational number.

4. Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$ (b) $\displaystyle\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$ (c) $\displaystyle\lim_{n\to\infty} \frac{1}{6n^2+1} = 0$

5. Determine the value of the following limits, and then prove your claims using the definition of convergence of a sequence.

   (a) $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1}$ (b) $\displaystyle\lim_{n\to\infty} \frac{4n+3}{7n-5}$ (c) $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n^{1/2}}$