Math 3100 **Spring 2018**

Exam 3
Study Guide and Practice Questions

1. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$\lim_{h\to 0}\big(f(x+h) - f(x-h)\big) = 0.$$
Does this imply that f is continuous at x?

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at x and satisfies
$$\lim_{h\to 0}\frac{f(x+h) - f(x-h)}{2h} = 0.$$
Does this imply that f is differentiable at x?

2. Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$.

   (a) Carefully state the ε-δ definition of $\lim_{x\to c} f(x) = L$.

   (b) Prove that $\lim_{x\to x_0} f(x) = L$ if and only if $\lim_{n\to\infty} f(x_n) = L$ for all sequences $\{x_n\}$ in $\mathbb{R} \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$.

3. Let
$$f_a(x) = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

   (a) For which values of a is f_a continuous at 0?

   (b) For which values of a is f_a differentiable at 0? In this case is the derivative function continuous?

   (c) For which values of a is f_a twice-differentiable?

4. Let
$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
Find particular non-negative (and potentially non-integer) values of a for which:

   (a) g_a is differentiable on $\mathbb{R}$, but g_a' is unbounded on $[0, 1]$.

   (b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at 0.

   (c) g_a and g_a' are differentiable on $\mathbb{R}$, but g_a'' is not continuous at 0.

5. (a) State and prove *Rolle's Theorem.*

   (b) State the *Generalized Mean Value Theorem.* Do you remember it's proof?

   (c) Let f and g be continuous functions on some interval I containing c and differentiable on the same interval except possibly at c itself. Prove that if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ and $\lim_{x\to c}\frac{f'(x)}{g'(x)} = L$, then
$$\lim_{x\to c}\frac{f(x)}{g(x)} = L.$$

6. (a) Find the value of $f^{(17)}(0)$ if $f(x) = \dfrac{4x}{2-x}$.

   (b) Give an example of an infinitely differentiable function that is <u>not</u> equal to its Taylor series.

   (c) i. Prove that if $h : [0, \infty) \to \mathbb{R}$ is twice differentiable on $[0, x]$, then
$$h(x) = h(0) + h'(0)x + \frac{h''(c)}{2}x^2$$
for some $c \in (0, x)$. *Hint: Apply the "Generalized MVT" to* $h(x) - h(0) - h'(0)x$ *and* x^2.

   ii. How well does $1 + x/2$ approximate $\sqrt{1+x}$ on $[0, 1/10]$?