Math 3100 **Spring 2018**

Exam 1

No calculators. Show your work. Give full explanations. Good luck!

1. (12 points) Determine which of the following sequences converge and which diverge. Find the value of the limit for those which converge.

   Be sure to give a short justification in each case by indicating any limit laws, theorems, or special limits used.

   (a) $a_n = (-1)^n n$

   (b) $b_n = \dfrac{2^n}{n^2 + 2^n}$

   (c) $c_n = (-1)^n - \dfrac{1}{n}$

   (d) $d_n = \sqrt{n+1} - \sqrt{n}$

2. (14 points)

   (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of $\lim_{n\to\infty} x_n = L$.

   (b) Use your definition above to prove that $\lim_{n\to\infty} \dfrac{2n+1}{n+2} = 2$.

   (c) Use your definition above to prove that if $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}$ is bounded (but not necessarily convergent), then $\lim_{n\to\infty} x_n y_n = 0$.

3. (12 points) Let $x_1 = 0$ and $x_{n+1} = \dfrac{2x_n + 1}{x_n + 2}$ for all $n \in \mathbb{N}$.

   (a) Find x_2, x_3, and x_4.

   (b) Show that $\{x_n\}$ converges and find the value of its limit.

4. (12 points)

   (a) Let $A \subseteq \mathbb{R}$. Prove that if $\sup(A)$ exists, then there exists a sequence $\{a_n\}$ of points in A with $\lim_{n\to\infty} a_n = \sup(A)$.

   (b) Let $\{b_n\}$ be a sequence of real numbers. Prove that

   $$\limsup_{n\to\infty} \frac{1}{1+b_n^2} \le 1.$$

   Clearly state your definition of $\limsup_{n\to\infty} x_n$ *and indicating why it exists when* $x_n = \dfrac{1}{1+b_n^2}$ *before proving that it cannot exceed* 1.