Math 3100 **Spring 2017**

Practice Exam 3

No calculators. Show your work. Give full explanations. Good luck!

1. (10 points)

   (a) Carefully state the definition of uniform convergence of a sequence of functions $\{f_n\}$ to a function f on a set A.

   (b) Consider the sequence of functions
   $$f_n(x) = \frac{x+n}{n}$$
   i. Find the pointwise limit of $\{f_n\}$ on $\mathbb{R}$.
   ii. Show that $\{f_n\}$ does not converge uniformly on $\mathbb{R}$.
   iii. Show that $\{f_n\}$ does converge uniformly on $[-M, M]$ for any $M > 0$.

2. (8 points) Consider the sequence of functions
   $$g_n(x) = \frac{x}{1+x^n}.$$
   (a) Find the pointwise limit of $\{g_n\}$ on $[0, \infty)$.
   (b) Explain how we know that the convergence cannot be uniform on $[0, \infty)$.
   (c) Write down a smaller set over which the convergence is uniform, no proofs required.

3. (10 points)

   (a) Prove that if $\sum_{n=0}^{\infty} h_n(x)$ converges uniformly on a set A, then the sequence of functions $\{h_n\}$ must converge uniformly to 0 on A.

   (b) Let
   $$h(x) = \sum_{n=0}^{\infty} \frac{1}{1+n^2x}.$$
   i. Prove that the series defining h does not converge uniformly on $(0, \infty)$.
   ii. Prove that h is however a continuous function on $(0, \infty)$.

4. (8 points) Evaluate the following infinite series
   $$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n}{4^n} \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}$$

5. (6 points)

   (a) Find the 4th order Maclaurin polynomial for $f(x) = \dfrac{\cos(x^2)}{1+x}$.

   (b) Use part (a) to find the value of $f^{(4)}(0)$ without differentiating.

6. (8 points)

   (a) Carefully state *Taylor's Theorem* (with Lagrangian Remainder) for Maclaurin Series.

   (b) Use the *Taylor's Theorem* to determine the following:
   i. An estimate for the accuracy of approximating $\sin x$ by $x - x^3/6$ when $|x| \leq 1/2$.
   ii. Values of x for which the accuracy of approximating $\sin x$ by $x - x^3/6$ is less than 10^{-3}.

   (c) Obtain an estimate for the accuracy of approximating
   $$\int_0^1 \frac{\sin x}{x}\, dx \quad \text{by} \quad 1 - \frac{1}{18}.$$