Math 3100 **Spring 2016**

Exam 1

No calculators. Show your work. Give full explanations. Good luck!

1. (8 points) Give counterexamples to the following **false** statements, no proofs are required.

   Note that in each instance the converse statement is in fact true.

   (a) If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent.

   (b) If $\{x_n\}$ is convergent, then $\{x_n\}$ is both bounded and monotone.

   (c) If $\{x_n\}$ contains a convergent subsequence, then $\{x_n\}$ is bounded.

   (d) If A contains its supremum, then A has finitely many elements.

2. (7 points) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers. Prove that if $\lim_{n\to\infty} x_n = x$ and $|x_n - y_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} y_n = x$.

3. (8 points)

   (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of $\lim_{n\to\infty} x_n = x$.

   (b) Use your definition to prove that $\lim_{n\to\infty} \frac{3n+4}{n+1} = 3$.

4. (7 points)

   (a) Carefully state the definition of a sequence of real numbers $\{x_n\}$ being a Cauchy sequence.

   (b) Give an example of a Cauchy sequence and prove that Cauchy sequences are always bounded.

5. (10 points)

   (a) Carefully state the *Axiom of Completeness* (the least upper bound axiom).

   (b) Let $\{x_n\}$ be a bounded increasing sequence of real numbers. Use the *Axiom of Completeness* to prove that $\lim_{n\to\infty} x_n$ exists and equals $\sup\{x_n : n \in \mathbb{N}\}$.

6. (10 points) Let $\{x_n\}$ be a bounded sequence of real numbers.

   (a) Carefully state the definition of $\limsup_{n\to\infty} x_n$ and justify why it always exists for such sequences.

   (b) Prove that if $|x_n| \leq 10$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} x_n \leq 10$.