Math 3100 **Spring 2017**

Exam 1 – Study Guide and Practice Questions

1. Form the logical negation of each of the following claims (all of which we actually know to be true, right?). One way to do this is to simply add "It is not the case that..." in front of each assertion, but for each statement, try and embed the word "not" as deeply into the resulting sentence as possible (or avoid using it altogether).

   (a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbb{N}$ such that $a + 1/n < b$.

   (b) Between every two distinct real numbers, there is an irrational number.

   (c) Given any real number x, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

2. (Reverse Triangle Inequality). Use the triangle inequality to prove that if $x, y \in \mathbb{R}$, then

$$\big||x| - |y|\big| \leq |x - y|.$$

3. Let $q \neq 0$ be rational and x be irrational. Prove that $q + x$ and qx are both irrational.

4. What happens if we reverse the order of the quantifiers in the definition of convergence of a sequence?

   Definition: A sequence $\{a_n\}$ *verconges* to a if *there exists* an $\varepsilon > 0$ such that *for all* N it is true that $n > N$ implies $|a_n - a| < \varepsilon$.

   Give an example of a vercongent sequence. Can you give an example a vercongent sequence that is divergent? What exactly is being described in this strange definition?

5. Let $\{x_n\}$ denote a sequence of reals and A denote a subset of $\mathbb{R}$.

   Carefully state the definition of the following:

   (a) A is bounded

   (b) $\sup(A)$ and $\inf(A)$

   (c) $\{x_n\}$ is bounded

   (d) $\{x_n\}$ converges to some real number x

   (e) $\{x_n\}$ diverges to infinity

   (f) a subsequence of $\{x_n\}$

   (g) $\{x_n\}$ is Cauchy

6. Use the definition of convergence to prove the following:

   (a) $\displaystyle\lim_{n\to\infty} \frac{3n+5}{n+3} = 3$

   (b) $\displaystyle\lim_{n\to\infty} \frac{2n+1}{2-n} = -2$

7. (a) Let $\{x_n\}$ be defined recursively by $x_1 = 3$ and $x_{n+1} = \dfrac{3x_n + 2}{x_n + 2}$ for each $n \in \mathbb{N}$. Show that $\{x_n\}$ converges and find its limit. What if $x_1 = 1$?

   (b) Let $\{x_n\}$ be defined by $x_n = \dfrac{1}{1+n} + \dfrac{1}{2+n} + \cdots + \dfrac{1}{n+n}$ for each $n \in \mathbb{N}$. Show that $\{x_n\}$ converges and find its limit.

   (c) Prove that if $x_n = 1 + \dfrac{1}{4} + \cdots + \dfrac{1}{n^2}$ for each $n \in \mathbb{N}$, then $\displaystyle\lim_{n\to\infty} x_n$ exists.

   Hint: First use induction to show that $x_n \leq 2 - \dfrac{1}{n}$ *for all* $n \in \mathbb{N}$.

8. Let $\{x_n\}$ denote a sequence of real numbers which converges to x, with $x > 0$. Use the definition of convergence to prove the following:
   (a) $\lim_{n\to\infty} |x_n| = |x|$
   (b) $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$
   (c) there exists $N \in \mathbb{N}$ such that $x_n > x/2$ for all $n \geq N$
   (d) $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$
   (e) $\{x_n\}$ is bounded
   (f) $\{x_n\}$ is Cauchy
9. Prove that if $|a_n| \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.
10. Carefully state the following:
    (a) The Axiom of Completeness (AoC)
    (b) The Archimedean Property
    (c) The Nested Interval Property (NIP)
    (d) The Monotone Convergence Theorem (MCT)
    (e) The Bolzano-Weierstrass Theorem (BW)
    (f) The Cauchy Criterion (CC)
11. Prove the following implications:
    (a) AoC $\Longrightarrow$ Archimedean Property
    (b) Archimedean Property $\Longrightarrow$ $\mathbb{Q}$ is dense in $\mathbb{R}$
    (c) AoC $\Longrightarrow$ NIP
    (d) AoC $\Longrightarrow$ MCT
    (e) MCT & "Rising Sun Lemma" $\Longrightarrow$ BW
    (f) BW $\Longrightarrow$ CC (you should also show that Cauchy $\Longrightarrow$ Bounded)
12. Let $A, B \subseteq \mathbb{R}$ which are non-empty, bounded above.
    (a) Show that if $A \subseteq B$, then $\sup A \leq \sup B$.
    (b) Show that if $\sup A < \sup B$, then there exists a $b \in B$ that is an upper bound for A.
13. Let $\{x_n\}$ denote a bounded sequence of real numbers.
    (a) Give the definition of $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$
    (b) Prove that there exists a subsequence of $\{x_n\}$ that converges to $\limsup_{n\to\infty} x_n$
    (c) Prove that if $\{y_n\}$ is a convergent subsequence of $\{x_n\}$, then its limit must be less than or equal to $\limsup_{n\to\infty} x_n$