Math 3100 **Spring 2017**

Exam 1

No calculators. Show your work. Give full explanations. Good luck!

1. (25 points)

   (a) Let $\{x_n\}$ be a sequence of real numbers. Carefully state the definition of the following:

   i. $\lim_{n\to\infty} x_n = x$

   ii. $\lim_{n\to\infty} x_n = \infty$.

   (b) Use the definition given in (i) to prove that $\lim_{n\to\infty} \frac{2n+1}{n-3} = 2$.

   (c) Use the definitions given above to prove that if $\lim_{n\to\infty} x_n = 2$, then

   i. $\{x_n\}$ is bounded

   ii. $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{2}$

   iii. $\lim_{n\to\infty} (x_n + y_n) = \infty$ whenever $\lim_{n\to\infty} y_n = \infty$

2. (12 points)

   (a) Carefully state the *Monotone Convergence Theorem*.

   (b) Let $x_1 = 1$ and $x_{n+1} = \left(\frac{n}{n+1}\right) x_n^2$ for all $n \in \mathbb{N}$.

   i. Find x_2, x_3, and x_4.

   ii. Show that $\{x_n\}$ converges and find the value of its limit.

3. (13 points) Let $\{x_n\}$ be a bounded sequence of real numbers.

   (a) Carefully state the definition of $\limsup_{n\to\infty} x_n$ and justify why it always exists for such sequences.

   (b) Prove that if $\alpha = \limsup_{n\to\infty} x_n$ and $\beta > \alpha$, then there exists an N such that $x_n < \beta$ whenever $n > N$.

   (c) Let $S = \{x :$ there exists a subsequence of $\{x_n\}$ that converges to $x\}$.

   i. Why do we know that S is non-empty?

   ii. Prove that if $x \in S$, then $x \leq \limsup_{n\to\infty} x_n$.