Math 3100 Assignment 4

Sequences and Completeness

Due at 1:00 pm on Monday the 17th of September 2018

1. Evaluate following limits or explain why they do not exist. Be sure to justify your answer.

   (a) $\lim_{n\to\infty} \left(\frac{2n+1}{3-n}\right)^3$

   (b) $\lim_{n\to\infty} \left((-1)^n + \frac{1}{n}\right)$

   (c) $\lim_{n\to\infty} \frac{\cos(n)}{n^2}$

   (d) $\lim_{n\to\infty} \frac{n! + n}{2^n + 3n!}$

   (e) $\lim_{n\to\infty} \frac{n + \log(n)}{n+1}$

2. (a) Prove that if $\{a_n\}$ is decreasing, then every subsequence of $\{a_n\}$ is also decreasing.

   (b) Let $\{x_n\}$ be a sequence of real numbers.

   Prove that $\{x_n\}$ contains a subsequence converging to x <u>if and only if</u> for all $\varepsilon > 0$ there exist infinitely many terms from $\{x_n\}$ that satisfy $|x_n - x| < \varepsilon$.

3. Let $A, B \subseteq \mathbb{R}$ which are non-empty, bounded above.

   (a) Show, directly from the definition of infimum, that if $A \subseteq B$, then $\inf(B) \leq \inf(A)$.

   (b) Show that if $\inf(B) < \inf(A)$, then there must exist $b \in B$ that is an lower bound for A.

   (c) Prove that there exists a sequence $\{a_n\}$ of points in A such that $\lim_{n\to\infty} a_n = \inf(A)$.

4. Let $x_1 = 0$ and $x_{n+1} = \frac{2x_n + 1}{x_n + 2}$ for all $n \in \mathbb{N}$.

   (a) Find x_2, x_3, and x_4.

   (b) Prove that $\{x_n\}$ converges and find the value of its limit.

   Hint: Try to apply the Monotone Convergence Theorem

5. Let $\{x_n\}$ be a bounded sequence of real numbers.

   (a) Prove that the "supremum of the tails of $\{x_n\}$" defined by $y_n := \sup\{x_n, x_{n+1}, \ldots\}$ is a decreasing sequence that is bounded below and conclude, using the Monotone Convergence Theorem, that $\{y_n\}$ is convergent.

   The value of $\lim_{n\to\infty} y_n$ is called the *limit superior* of $\{x_n\}$ is usually denoted by $\limsup_{n\to\infty} x_n$.

   (b) Prove that if $\alpha = \limsup_{n\to\infty} x_n$ and $\beta > \alpha$, then there exists an N such that $x_n < \beta$ whenever $n > N$.