Preface

This book grew out of a set of notes from a graduate course in which I tried to introduce the students, in a single course, to both algebraic number theory and the theory of curves. Most books in the literature that touch upon both subjects discuss the function field case from a very algebraic point of view. The modern language of arithmetic geometry is very geometric, and I hope that it will be useful to have in the literature a book such as this with a more geometric introductory presentation of the function field case.

Let us define arithmetic geometry in this preface to be the study of the solutions in k^n of a system of polynomial equations in n variables with coefficients in a ring k (such as $k = \mathbb{Z}$, $k = \mathbb{Q}$, or $k = \mathbb{Z}/p\mathbb{Z}$). While the central problem of arithmetic geometry is thus easily described, mastering the powerful tools developed in the last thirty years to study solutions of polynomials is extremely challenging for beginners[1]. A student with a basic knowledge of algebra and Galois theory will first have to take a course in algebraic number theory, a course in commutative algebra, and a course in algebraic geometry (including scheme theory) to be able to understand the language in which theorems and proofs are stated in modern arithmetic geometry. Moreover, a student who has had the opportunity to take those three courses will be faced with the additional hurdle of understanding their interconnections.

An Invitation to Arithmetic Geometry tries to present in a unified manner, from the beginning, some of the basic tools and concepts in number theory, commutative algebra, and algebraic geometry, and to bring out the deep analogies between these topics. This book introduces the reader to arithmetic geometry by focusing primarily on the dimension one case (that is, curves in algebraic geometry, rings of dimension one in commutative algebra). Topics covered and interconnected include:

1. rings of integers, discriminant and ramification, ideal class group (in algebraic number theory),
2. localizations, Dedekind domains, discrete valuation rings (in commutative algebra),
3. affine and projective curves, the Riemann-Roch theorem (in algebraic geometry), and

[1] An example of a polynomial equation in two variables is the Fermat equation $x^n + y^n = 1$. The celebrated Fermat's Last Theorem states that the only solutions in rational numbers to the equation $x^n + y^n = 1$ are the "obvious" ones if $n > 2$. Ribet's proof, in 1986, that Fermat's Last Theorem holds if a weak form of the conjecture of Shimura-Tanyiama-Weil holds is an example of modern tools in arithmetic geometry producing new and deep results about easily stated problems in number theory. Wiles announced on June 23, 1993, that he can prove this weak form of the Shimura-Tanyiama-Weil conjecture and, therefore, that Fermat's Last Theorem holds [?], [?].

4. the zeta-function of a curve over a finite field and the analogue of the Riemann hypothesis (in arithmetic geometry).

An example of a fundamental analogy relating these fields of study is the analogy between a ring of integers with its set of archimedian absolute values in number theory, and its analogue in algebraic geometry, an affine curve with its set of points at infinity. This analogy is key to many of the recent translations of statements made in number theory to statements in algebraic geometry, and vice-versa. Unfortunately, this analogy is usually not developed in a course in number theory, nor in a course in algebraic geometry. The present text draws out this and other underlying analogies between these fields.

An Invitation to Arithmetic Geometry is designed as a textbook for a year-long introduction to arithmetic geometry. It includes extensive examples to illustrate each new concept and contains problems at the end of each chapter to help the student grasp the material presented. Most of the results presented in this book are classical and, apart from Bombieri's 1972 proof of the Riemann hypothesis for curves over finite fields, have been in the literature since the 1940s. I have tried to include short historical remarks whenever possible. This book is not designed to be encyclopedic. It introduces new concepts as they are needed and, where possible, uses this need for a concept to motivate its introduction. Our choice to discuss only the case of dimension one means that certain theorems in the text are not stated in their most general form, but only in the form needed for the purposes of the book. When a theorem is not stated in its most general form, references for generalizations of the theorem are included.

While this book is not meant to be an introduction to the theory of schemes, *per se*, I have tried to indicate in the text how the geometric notions introduced relate to schemes (see, e.g., ??.??, ??.??, ??.??, and ??.??). The field of arithmetic geometry has developed tremendously in the last thirty years, and several major open problems have recently been solved, such as Mordell's conjecture ??.?? and Fermat's Last Theorem. The proofs of these deep theorems use the language of schemes and the techniques of algebraic geometry. A student of arithmetic geometry needs to master these difficult techniques in addition to the materials presented in this book. I hope that the unified introduction to these topics presented in this book will serve as a motivation for students to learn these techniques and pursue further study of the beautiful interconnections between arithmetic and geometry.

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