On Ext *and exact sequences.*

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§0. Introduction.

Since H. Cartan and S. Eilenberg have introduced the theory of the functors Ext_Λ^n and satellites of functors [4], it has found great many applications, and various generalizations have been successful. Among other generalizations ([5, §7], [6], [8], [9, VII]), Buchsbaum [1] introduced the notions of exact categories, and showed that the domain of the theory is readily extended from the category of Λ-modules to exact categories with sufficiently many projectives or injectives. A. Heller [7] has further extended the domain to what he calls 'abelian categories' with enough projectives or injectives. Now a paper by the author [10], giving a 1-1-correspondence between $\mathrm{Ext}_\Lambda^n(A, B)$ and the collection of n-fold extensions over A with kernel B properly classified, was suggesting generalizations in another direction, namely to avoid the use of projectives or injectives, hence extending the domain of the theory to a much wider class of additive categories. This is what we aim at in the present paper. In a similar direction we note the independent works of Buchsbaum [2], [3]. It is perhaps worth mentioning the confusion that independently of Heller [7] many mathematicians say 'abelian categories' to mean exact categories with the axiom of existence of direct sums, with or without projectives and/or injectives (cf. [6]). In this paper we shall not follow the terminology in [7] hoping that the reader of [7] will not get confused.

In trying to classify exact sequences in an additive category so as to get a good theory of Ext and satellites, one will find that imbeddings of an exact sequence $0 \to B \to E \to A \to 0$ and maps $A' \to A$, $B \to B''$ in commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & E' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0, \end{tikzcd} \quad (\Rightarrow:\ \text{identity map}) \quad \begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, Rightarrow] & 0 \\ 0 \arrow[r] & B'' \arrow[r] & E'' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

with exact rows are the fundamental media. After the preparatory §1, we shall investigate in §2 what are the consequences of these imbeddings and under what circumstances these imbeddings are possible, which will make our motivation of introducing the notions of regular S-categories and quasi-abelian S-categories.

The idea in §3 can be explained as follows: The n-fold extensions over A with kernel B in an additive category $\mathcal{A}$ will be considered as some quantity lying

between A and B, or lying over the pair (A, B), which we want to classify to get $\mathrm{Ext}^n(A, B)$. Then the totality of n-fold extensions in $\mathcal{A}$ is considered as a sort of web spanning pairs of objects. We have the notions of maps of n-fold extensions, which make the web a category over the pair $(\mathcal{A}, \mathcal{A})$. In other words the web gives a certain correspondence between $\mathcal{A}$ and a copy of $\mathcal{A}$ with functorial multiplicity, which renders the functorial structure of Ext^n. In generalizing this situation we consider a pair of categories $(\mathcal{A}, \mathcal{B})$ and a third category $\mathcal{X}$ together with two covariant functors $S_-: \mathcal{X}\to\mathcal{A}$, $S_+: \mathcal{X}\to\mathcal{B}$, or to the same effect, a covariant functor $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$, which we call a span over $(\mathcal{A}, \mathcal{B})$. We then show that under certain conditions the spanning objects can be classified giving rise to a functor $\bar{S}=\bar{S}(a, b)$ of two variables, a contravariant in $\mathcal{A}$, and b covariant in $\mathcal{B}$. The result is soon applied to the span of n-fold extensions, and we get functors Ext^n. Further the composition product in Ext as defined in [10] will be presented in a generalized form in terms of spans.

Let M, M' be functors of a category C with values in the category $\mathcal{M}$ of additive groups and homomorphism. When M, M' have the same variance, we have the additive group $\mathrm{Hom}_C(M, M')$ of natural transformations $M\to M'$. As an operation adjoint to Hom_C we shall present in §4 the tensor product $M\otimes_C M'$. This is an additive group defined when M is contravariant and M' is covariant. Formal properties for Hom_Λ and $\otimes_\Lambda$ have remarkable analogues for Hom_C and $\otimes_C$. Satellites of functors originally defined using projectives or injectives will then be redefined, without recourse to projectives or injectives, but in terms of Hom_C, $\otimes_C$, and Ext_C. This gives us for example the interpretation that $\mathrm{Tor}_n{}^\Lambda(A, B)$ is the group of natural transformations from the covariant functor $\mathrm{Ext}_\Lambda{}^n(A, x)$ to the covariant functor $x\otimes_\Lambda B$.

In the last §5 we shall develop a study on the nature of 'similarity' of exact sequences. In the course of classification of n-fold extensions, we first introduce a binary relation $\rightrightarrows$ which is reflexive and transitive, but not necessarily symmetric. Then we define the similarity $\sim$ as the equivalence relation generated by $\rightrightarrows$. Thus $\sim$ involves a series of relations $\rightrightarrows$, $\leftleftarrows$, of which the number may not be uniformly bounded. In the classification of 1-fold extensions, i.e. short exact sequences, $\rightrightarrows$ is already symmetric, and the similarity classification is in a sense faithful. We shall exhibit certain connections among similar n-fold extensions, which will show that similar n-fold extensions are not too far away from each other.

In order to facilitate maneuvering maps and diagrams in a category, we shall introduce the following convention: A map $A\to B$ once written with an arrow in

the context, whether by itself or in a diagram, shall be designated by AB. The composition of two (or more) maps $A \to B$, $B \to C$ shall be denoted by $AB \cdot BC$ ($AB \cdot BC \cdot CD \cdots$) or by ABC ($ABCD \cdots$). We shall not write AC for ABC unless having written a direct arrow $A \to C$ saying $AC = ABC$. Otherwise AC shall denote another map $A \to C$ defined in the context. If we want to respect the name given to a map $\varphi: A \to B$, or if we have to distinguish one from several maps from A to B all expressed by direct arrows $A \to B$, then we shall write $A\varphi B$, $A\varphi' B$, etc. In this notation $A\varphi B$, the same letter φ may be used to denote other maps of apparently different domains and ranges. For example $\varphi_\nu: A_\nu \to B_\nu$ ($\nu = 1, 2$) may be denoted as $A_\nu \varphi B_\nu$. The identity map $e_A: A \to A$ will be written as $A \Rightarrow A$ in diagrams and as AeA in expressions following this convention. AA may denote some other map defined in the context.

The above convention is a slightly modified version of that introduced in [10]. It has certain advantages and disadvantages, and will be obeyed when it is more convenient than the usual convention of naming maps as $\varphi: A \to B$ and writing down the composition of maps as $\psi \circ \varphi$. The latter will be obeyed as well.

Our domain of theory will be abstract categories, and no applications are intended in this paper. They will be found elsewhere. Also metamathematical preoccupations, such as whether a certain collection is a set or not, are put aside. Upper asterisques * attached to a category or to a categorical statement will mean the dual category or the dual statement (cf. [1]).

§1. Additive categories.

1.0. A *preadditive* category by definition is a category $\mathcal{A}$ in which addition is defined in each collection $\mathrm{Hom}(A, B)$ of maps $A \to B$, subject to the two axioms:

(A1) For any pair of objects A, B the collection $\mathrm{Hom}(A, B)$ is not empty and makes an additive group.

(A2) Composition is bilinear, i.e. for $\varphi_1, \varphi_2 = \mathrm{Hom}(A, B)$, $\psi_1, \psi_2 = \mathrm{Hom}(B, C)$ we have $(\psi_1 + \psi_2) \circ (\varphi_1 + \varphi_2) = \psi_1 \circ \varphi_1 + \psi_2 \circ \varphi_1 + \psi_1 \circ \varphi_2 + \psi_2 \circ \varphi_2$.

Without loss of generality we may assume that there is a unique object $\emptyset$ such that $\mathrm{Hom}(\emptyset, \emptyset) =$ zero group, or equivalently $e_\emptyset = 0$. If there are many such neutral objects, they are connected to each other by unique equivalence maps, and so can be unified to a single object. If there is no neutral object we can attach to $\mathcal{A}$ an object $\emptyset$ with $\mathrm{Hom}(A, \emptyset)$, $\mathrm{Hom}(\emptyset, A)$, $\mathrm{Hom}(\emptyset, \emptyset)$ defined to be trivial. Now that every object A has unique maps $A \to \emptyset$, $\emptyset \to A$ we may write $A\emptyset$, $\emptyset A$ without ambiguity. $A\emptyset B$ is the zero element in $\mathrm{Hom}(A, B)$, which we denote alternatively as 0^A_B or 0.

A map $\alpha: A' \to A$ in $\mathcal{A}$ induces a homomorphism

$$\mathrm{Hom}(\alpha, B): \ \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B)$$

sending φ to $\varphi\circ\alpha$. This homomorphism will be denoted also as $\circ\alpha$. Dually a map $\beta: B \to B'$ induces a homomorphism

$$\mathrm{Hom}(A, \beta): \ \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$$

sending φ to $\beta\circ\varphi$. This will be denoted also as $\beta\circ$. $\mathrm{Hom}(a, b)$ is a functor of two variables a contravariant in $\mathcal{A}$, b covariant in $\mathcal{A}$, and with values in the category $\mathcal{M}$ of additive groups and homomorphisms. It is *additive* in both variables:

$$\mathrm{Hom}(\alpha_1+\alpha_2, B) = \mathrm{Hom}(\alpha_1, B) + \mathrm{Hom}(\alpha_2, B),$$
$$\mathrm{Hom}(A, \beta_1+\beta_2) = \mathrm{Hom}(A, \beta_1) + \mathrm{Hom}(A, \beta_2).$$

In general by a *(left) $\mathcal{A}$-module* we mean a covariant additive functor $\mathcal{A} \to \mathcal{M}$, and by a *right $\mathcal{A}$-module* (or an *$\mathcal{A}^*$-module*) we mean a contravariant additive functor $\mathcal{A} \to \mathcal{M}$. An additive functor of several variables with values in $\mathcal{M}$ will accordingly be called $\mathcal{A}^*$-$\mathcal{B}$-module, $\mathcal{A}$-$\mathcal{B}^*$-$\mathcal{C}$-module, etc. The Hom functor of the preadditive category $\mathcal{A}$ is an $\mathcal{A}^*$-$\mathcal{A}$-module.

In the following paragraphs of this section $\mathcal{A}$ will stand for a preadditive category. Unless otherwise stated, objects and maps will mean those in $\mathcal{A}$.

1.1. A map $\varphi: A \to B$ is called an *injection* if $\mathrm{Hom}(C, \varphi)$ is a monomophism for every object C. This means that for any map $C \to A$, $CAB=0$ implies $CA=0$, or equivalently for any three maps $C \to A$, $C \to C'$, and $C' \to A$, $CAB=CC'AB$ implies $CA=CC'A$. Dually a map $\varphi: A \to B$ is called a *projection* if $\mathrm{Hom}(\varphi, C)$ is a monomorphism for every object C. An injection which is also a projection is called a *bijection*. By the *kernel* of a map $A \to B$ we mean a map $N \to A$ such that the induced sequence

$$0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$$

is exact for every object C. This means that (i) $NAB=0$, and that (ii) for any map $C \to A$ with $CAB=0$ there is a unique map $C \to N$ such that $CNA=CA$. The latter condition (ii) is equivalent to that for any map $C \to A$ with $CAB=0$ there exists a map $C \to N$ such that $CNA=CA$, and NA is an injection. If $B \to B'$ is an injection, then $N \to A$ is the kernel of AB if and only if it is the kernel of ABB'. This is immediate from the definition. If in the commutative diagram

$$\begin{array}{ccccc} N & \to & A & \to & B \\ & & \downarrow & & \downarrow \\ N' & \to & A' & \to & B' \end{array}$$

NA, $N'A'$ are the kernels of AB and $A'B'$ respectively, then there is a unique map $N \to N'$ with which the diagram remains commutative. If AA', BB' are equivalence maps, then NN' is necessarily an equivalence map. Thus the kernel

of a map $A \to B$ is unique upto equivalence. So we shall write $NA = \text{kernel}\, AB$ to mean that NA is the kernel of AB, and $(N \to A) = \text{kernel}\, AB$ to mean that AB has the kernal, and $N \to A$ is defined as the kernel of AB. The object N determined upto equivalence will be written as $N = \text{Ker}\, AB$.

Cokernels of maps are defined dually. Namely $A \to B$ has the cokernel $B \to L$ if $ABL = 0$ and if for any map $B \to C$ with $ABC = 0$ there is a unique map $L \to C$ such that $BLC = BC$. A map $A \to B$ is an injection (projection) if and only if kernel $AB = 0$ (cokernel $AB = 0$), i.e., $\text{Ker}\, AB = \emptyset$ $(\text{Coker}\, AB = \emptyset)$. If $AB = 0$, then $AeA = \text{kernel}\, AB$, and $BeB = \text{cokernel}\, AB$. As later examples will show not all maps may have kernels and cokernels. Even if all maps have kernels and cokernels, an injection may not be the kernel of any map, and a bijection may not be an equivalence map.

Given a class $\mathcal{S}$ of maps we shall denote by kernel $\mathcal{S}$ the totality of kernels of maps in $\mathcal{S}$, and by kernel$^{-1}\mathcal{S}$ the totality of maps which admit kernels in $\mathcal{S}$. Similarly cokernel $\mathcal{S}$, cokernel$^{-1}\mathcal{S}$ are defined. Also $\mathcal{S}_2 \circ \mathcal{S}_1$ will denote the totality of compositions $\varphi_2 \circ \varphi_1$ $(\varphi_1 \in \mathcal{S}_1, \varphi_2 \in \mathcal{S}_2)$. Suppose $A \to B$ belongs to kernel^{-1}(cokernel$^{-1}\mathcal{A}$), and put $(N \to A) = \text{Kernel}\, AB$, $(A \to M') = \text{cokernel}\, NA$. Then we have $NA = \text{kernel}\, AM'$. Suppose further AB is in cokernel^{-1} (kernel$^{-1}\mathcal{A}$), and put $(B \to L) = \text{cokernel}\, AB$, $(M \to B) = \text{kernel}\, BL$. Then there is a unique map $M' \to M$ such that $AM'MB = AB$. If $M'M$ is an equivalence map, the map AB is called *proper*. Thus a map $A \to B$ is proper if and only if it can be imbedded in the commutative diagram

$$\begin{tikzcd} & & M \arrow[dr] & & \\ & A \arrow[ur] \arrow[rr] & & B \arrow[dr] & \\ N \arrow[ur] & & & & L \end{tikzcd}$$

where $NA = \text{kernel}\, AM$, $AM = \text{cokernel}\, NA$, $BL = \text{cokernel}\, MB$, and $MB = \text{kernel}\, BL$. This imbedding is unique upto equivalence, and we shall write $AM = \text{coimage}\, AB$, $MB = \text{image}\, AB$, $M = \text{Im}\, AB$. The totality of proper maps (proper injections, proper projections) will be denoted by $\mathrm{P}\mathcal{A}$ $(\mathrm{PI}\mathcal{A}, \mathrm{PJ}\mathcal{A})$. A map is a proper injection if and only if it has the cokernel and is the kernel of its cokernel. Denote by $\mathrm{I}\mathcal{A}$, $\mathrm{J}\mathcal{A}$, $\mathrm{E}\mathcal{A}$ the totalities of injections, of projections, and of equivalence maps respectively. We have

$$\mathrm{P}\mathcal{A} \cap \mathrm{I}\mathcal{A} \cap \mathrm{J}\mathcal{A} = \mathrm{E}\mathcal{A},$$

$$\mathrm{P}\mathcal{A} = \mathrm{PI}\mathcal{A} \circ \mathrm{PJ}\mathcal{A},$$

$$\mathrm{PI}\mathcal{A} = \text{kernel cokernel}\, \mathcal{A} = \text{kernel}\, \mathcal{A} \cap \text{cokernel}^{-1}\, \mathcal{A},$$

$$\mathrm{PJ}\mathcal{A} = \text{cokernel kernel}\, \mathcal{A} = \text{cokernel}\, \mathcal{A} \cap \text{kernel}^{-1}\, \mathcal{A}.$$

An *exact category* [1] by definition is a preadditive category $\mathcal{A}$ in which every map is proper.

A sequence of maps in a preadditive category is called *exact* if every map in the sequence is proper, and if for any consecutive two maps $A \to B \to C$ in the sequence we have kernel BC = image AB, or equivalently cokernel AB = coimage BC. Thus exactness of $\emptyset \to A \to B$ means that AB is a proper injection, exactness of $B \to C \to \emptyset$ means that BC is a proper projection, exactness of $\emptyset \to A \to A' \to \emptyset$ means that AA' is an equivalence map, and exactness of $\emptyset \to B \to E \to A \to \emptyset$ means $BE =$ kernel EA, $EA =$ cokernel BE. If one exact sequence X starts from $\emptyset \to B \to E \to \cdots$, and another exact sequence Y terminates in $\cdots \to F \to B \to \emptyset \to$, then the *composed sequence*

$$Y \bigcirc X: \quad \cdots \to F \to E \to \cdots \qquad (FE = FBE)$$

is exact.

1.2. We denote by $\mathcal{E}^0\mathcal{A}$ the preadditive category given by the following data:

(i) An object X in $\mathcal{E}^0\mathcal{A}$ means a map $\varphi: A \to B$ in $\mathcal{A}$, i.e. a triple $X = \langle A, B, \varphi \in \mathrm{Hom}(A, B) \rangle$;

(ii) A map $\xi: X \to X'$ in $\mathcal{E}^0\mathcal{A}$ means a commutative diagram

$$\begin{array}{cccc} X: & A & \xrightarrow{\varphi} & B \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ X': & A' & \xrightarrow{\varphi'} & B' \end{array}$$

i.e. a quadruple $\xi = \langle \varphi, \varphi', \alpha \in \mathrm{Hom}(A, A'), \beta = \mathrm{Hom}(B, B') \rangle$ such that $\beta \circ \varphi = \varphi' \circ \alpha$;

(iii) Composition is given by composing the vertical constituents, namely

$$\langle \varphi', \varphi'', \alpha', \beta' \rangle \circ \langle \varphi, \varphi', \alpha, \beta \rangle = \langle \varphi, \varphi'', \alpha' \circ \alpha, \beta' \circ \beta \rangle;$$

(iv) Addition of maps $\varphi \to \varphi'$ is given by adding up the vertical constituents, namely

$$\langle \varphi, \varphi', \alpha_1, \beta_1 \rangle + \langle \varphi, \varphi', \alpha_2, \beta_2 \rangle = \langle \varphi, \varphi', \alpha_1 + \alpha_2, \beta_1 + \beta_2 \rangle.$$

This category is called the *category of maps* in $\mathcal{A}$. E^0_-, E^0_+ will denote the covariant additive functors $\mathcal{E}^0\mathcal{A} \to \mathcal{A}$ defined by

$$E^0_-\langle A, B, \varphi \rangle = A, \qquad E^0_-\langle \varphi, \varphi', \alpha, \beta \rangle = \alpha,$$
$$E^0_+\langle A, B, \varphi \rangle = B, \qquad E^0_+\langle \varphi, \varphi', \alpha, \beta \rangle = \beta.$$

In this category a map $\dot{X} \to X$ is the kernel of $X \to X'$ if and only if $E^0_k(\dot{X}X)$ is the kernel of $E^0_k(XX')$ $(k = -, +)$. If $\mathcal{A}$ is an exact categery, so is $\mathcal{E}^0\mathcal{A}$.

For an integer $n \geqq 1$ we denote by $\mathcal{E}^n\mathcal{A}$ the preadditive category given by the following data:

(i) An object X in $\mathcal{E}^n\mathcal{A}$ means an exact sequence in $\mathcal{A}$ of the form

$$X:\ 0 \to B \to E_n \to E_{n-1} \to \cdots \to E_1 \to A \to 0\,;$$

(ii) A map $\xi: X \to X'$ in $\mathcal{E}^n\mathcal{A}$ means a commutative diagram with exact rows in $\mathcal{A}$ of the form

$$\begin{array}{cccccccccccc} X: & 0 & \to & B & \to & E_n & \to \cdots \to & E_1 & \to & A & \to & 0 \\ & & & \downarrow{\scriptstyle\beta} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\ X': & 0 & \to & B' & \to & E'_n & \to \cdots \to & E'_1 & \to & A' & \to & 0\,; \end{array}$$

(iii) Composition is given by composing the vertical constituents;

(iv) Addition is given by adding up the vertical constituents.

$\mathcal{E}^n\mathcal{A}$ $(n \geqq 1)$ is called the *category of n-fold extensions* in $\mathcal{A}$, in particular $\mathcal{E}^1\mathcal{A}$ is called the *category of extensions* in $\mathcal{A}$ or the *category of short exact sequences* in $\mathcal{A}$. E^n_-, E^n_+ will denote the covariant additive functors $\mathcal{E}^n\mathcal{A} \to \mathcal{A}$ defined by

$$E^n_-(X)=A,\ E^n_-(\xi)=\alpha,\ E^n_+(X)=B,\ E^n_+(\xi)=\beta.$$

Also E^n_k $(k=0, 1, 2, \cdots, n+1)$ will stand for the covariant additive functors $\mathcal{E}^n\mathcal{A} \to \mathcal{A}$, $E^n_0=E^n_-$, $E^n_k(X)=E_k$, $E^n_k(XX')=E_kE'_k$ $(1 \leqq k \leqq n)$, $E^n_{n+1}=E^n_+$.

The situation about kernels and cokernels in $\mathcal{E}^n\mathcal{A}$ $(n \geqq 1)$ is not so simple as in $\mathcal{E}^0\mathcal{A}$. For illustration we shall consider the sequence

$$\begin{array}{cccccccccccc} \dot{X}: & 0 & \to & \dot{B} & \to & \dot{E}_n & \to \cdots \to & \dot{E}_1 & \to & \dot{A} & \to & 0 \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X: & 0 & \to & B & \to & E_n & \to \cdots \to & E_1 & \to & A & \to & 0 \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \bar{X}: & 0 & \to & \bar{B} & \to & \bar{E}_n & \to \cdots \to & \bar{E}_1 & \to & \bar{A} & \to & 0 \end{array}$$

in $\mathcal{E}^n\mathcal{A}$ $(n \geqq 1)$.

PROPOSITION 1.1. $\dot{X}X=$kernel $\bar{X}X$ *if and only if* $E^n_k(\dot{X}X)=$kernel $E^n_k(X\bar{X})$ *for* $k=1, \cdots, n, n+1$. $X\bar{X}=$cokernel $\dot{X}X$ *if and only if* $E^n_k(X\bar{X})=$cokernel $E^n_k(\dot{X}X)$ *for* $k=0, 1, \cdots, n$.

PROOF. In view of duality we prove the first part. For an object C we shall denote by C^n_k $(k=1, \cdots, n, n+1)$ the object in $\mathcal{E}^n\mathcal{A}$ having $C \Rightarrow C$ for $E^n_k(C^n_k) \to E^n_{k-1}(C^n_k)$ and 0's elsewhere. Then $\mathrm{Hom}(C^n_k, X)$ is naturally isomorphic to $\mathrm{Hom}(C, E^n_k(X))$. Therefore, if $\dot{X}X=$kernel $X\bar{X}$, then the sequence

$$0 \to \mathrm{Hom}(C^n_k, \dot{X}) \to \mathrm{Hom}(C^n_k, X) \to \mathrm{Hom}(C^n_k, \bar{X})$$

is exact, and so the sequence

$$0 \to \mathrm{Hom}(C, E^n_k(\dot{X})) \to \mathrm{Hom}(C, E^n_k(X)) \to \mathrm{Hom}(C, E^n_k(\bar{X}))$$

is also exact. This shows $E^n_k(\dot{X}X)=$kernel $E^n_k(X\bar{X})$ for $k=1, \cdots, n, n+1$. Now suppose the converse, and let

$$\begin{array}{cccccccccccc} X': & 0 & \to & B' & \to & E'_n & \to \cdots \to & E'_1 & \to & A' & \to & 0 \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ X: & 0 & \to & B & \to & E_n & \to \cdots \to & E_1 & \to & A & \to & 0 \end{array}$$

be a map in $\mathcal{E}^n\mathcal{A}$ with $X'X\bar{X}=0$. Then we have $E_1'E_1\bar{E}_1=0,\cdots,E_n'E_n\bar{E}_n=0$, $B'B\bar{B}=0$, and so there are uniques maps $E_1'\to\dot{E}_1,\cdots,E_n'\to\dot{E}_n$, $B'\to\dot{B}$ such that $E_1'\dot{E}_1E_1=E_1'E_1,\cdots,E_n'\dot{E}_nE_n=E_n'E_n$, $B'\dot{B}B=B'B$. Since we have $B'E_n'\dot{E}_n\cdot\dot{E}_nE_n=B'E_n'E_n=B'BE_n=B'\dot{B}BE_n=B'\dot{B}\dot{E}_n\cdot\dot{E}_nE_n,\cdots,E_2'E_1'\dot{E}_1\cdot\dot{E}_1E_1=E_2'E_1'E_1=E_2'E_2E_1=E_2'\dot{E}_2E_2E_1=E_2'\dot{E}_2\dot{E}_1\cdot\dot{E}_1E_1$, commutativity holds in the diagram

$$\begin{array}{lccccccccccccc}
X': & \emptyset & \to & B' & \to & E_n' & \to\cdots\to & E_2' & \to & E_1' & \to & A' & \to & \emptyset \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
\dot{X}: & \emptyset & \to & \dot{B} & \to & \dot{E}_n & \to\cdots\to & \dot{E}_2 & \to & \dot{E}_1 & \to & \dot{A} & \to & \emptyset\,.
\end{array}$$

So there is a unique map $A'\to\dot{A}$ completing a map $X'\to\dot{X}$ in $\mathcal{E}^n\mathcal{A}$. From $E_1'A'\cdot A'\dot{A}A=E_1'\dot{E}_1\dot{A}A=E_1'\dot{E}_1E_1A=E_1'E_1A=E_1'A'\cdot A'A$ follows $A'\dot{A}A=A'A$, and so $X'\dot{X}$ is the unique map such that $X'\dot{X}X=X'X$, q.e.d.

From Proposition 1.1, we see that for the map $X\to\bar{X}$ in $\mathcal{E}^n\mathcal{A}$ to admit the kernel it is necessary and sufficient that every vertical constituent except for $A\to\bar{A}$ has the kernel and that the kernel sequence $\emptyset\to\operatorname{Ker}B\bar{B}\to\operatorname{Ker}E_n\bar{E}_n\to\cdots\to\operatorname{Ker}E_1\bar{E}_1$ is exact.

In particular the map in $\mathcal{E}^n\mathcal{A}$ ($n\geqq 3$) given by the diagram

$$\begin{array}{lccccccccccccccc}
C_n^n: & \emptyset & \to & \emptyset & \to & C & \Rightarrow & C & \to & \emptyset & \to\cdots\to & \emptyset & \to & \emptyset \\
 & & & \downarrow & & \Downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_{n+1}^n: & \emptyset & \to & C & \Rightarrow & C & \to & \emptyset & \to & \emptyset & \to\cdots\to & \emptyset & \to & \emptyset
\end{array}$$

admit no kernel unless $C=\emptyset$. If $X\to\bar{X}$ is a bijection in $\mathcal{E}^n\mathcal{A}$ ($n\geqq 2$), then kernel $X\bar{X}=0$, cokernel $X\bar{X}=0$, and so kernel $E_k^n(X\bar{X})=0$, cokernel $E_k^n(X\bar{X})=0$ for $k=1,\cdots,n$. Therefore if $\mathcal{A}$ is an exact category, a bijection in $\mathcal{E}^n\mathcal{A}$ ($n\geqq 2$) is necessarily an equivalence map. In $\mathcal{E}^1\mathcal{A}$ and in $\mathcal{E}^2\mathcal{A}$ we see easily that all maps have kernels and cokernels if $\mathcal{A}$ is an exact category. Thus $\mathcal{E}^2\mathcal{A}$ is an exact category if $\mathcal{A}$ is an exact category. However, in $\mathcal{E}^1\mathcal{A}$ we have the example of a bijection which is not an equivalence map, as provided by the left part of the above diagram.

Another example is presented by the category $\mathcal{C}_A$ of commutative algebraic groups [9, VIII]. This category is not exact unless the characteristic is zero.

1.3. By the *direct sum* of two objects A_1, A_2 we mean an object A together with four maps $A_1\to A$, $A_2\to A$, $A\to A_1$, $A\to A_2$ such that

$$A_1AA_1=A_1eA_1,\quad A_2AA_2=A_2eA_2,\quad AA_1A+AA_2A=AeA.$$

In this case we shall write

$$(A_1A,\ A_2A,\ AA_1,\ AA_2):\quad A\approx A_1\oplus A_2.$$

Note that the above three identities imply $A_1AA_2=0$, $A_2AA_1=0$. The sequences

$$\emptyset\to A_1\to A\to A_2\to\emptyset,\qquad \emptyset\to A_2\to A\to A_1\to\emptyset$$

are obviously exact.

If we have $(B_1 \to B, B_2 \to B, B \to B_1, B \to B_2): B \approx B_1 \oplus B_2$, then for each pair of maps $A_1 \to B_1$, $A_2 \to B_2$ there is a unique map $A \to B$ such that

$$A_\nu B_\nu B = A_\nu AB, \quad ABB_\nu = AA_\nu B_\nu \ (\nu = 1, 2).$$

If further A_1B_1, A_2B_2 are equivalence maps, then AB is also an equivalence map. In particular the direct sum of two objects is unique upto equivalence. An *additive* (*abelian*) *category* by definition is a preadditive (exact) category satisfying the axiom:

(A3) Any pair of objects has the direct sum.

In the sequel $A_1 \oplus A_2$ will stand for the direct sum A of A_1 and A_2 fixed once and for all with the four maps $A_1 \to A$, $A_2 \to A$, $A \to A_1$, $A \to A_2$ also fixed once and for all. These four maps are called the *canonical maps* for $A_1 \oplus A_2$, and will be denoted by $A_1c(A_1 \oplus A_2)$, $A_2c(A_1 \oplus A_2)$, $(A_1 \oplus A_2)\mathfrak{d}A_1$, $(A_1 \oplus A_2)\mathfrak{d}A_2$. The first two are the *canonical injections* and the last two are the *canonical projections*. For fear of confusion one may write $Ac_1(A \oplus A)$, $Ac_2(A \oplus A)$ according as it refers to the first component or the second. The obvious canonical equivalence maps $A_1 \oplus A_2 \to A_2 \oplus A_1$, $(A_1 \oplus A_2) \oplus A_3 \to A_1 \oplus (A_2 \oplus A_3)$ will be designated with insertion of the symbol $\mathfrak{d}c$. For simplicity the direct sums $\emptyset \oplus A$, $A \oplus \emptyset$ shall be those given, by

$$(\emptyset A, AeA, A\emptyset, AeA): \ A \approx \emptyset \oplus A,$$
$$(AeA, \emptyset A, AeA, A\emptyset): \ A \approx A \oplus \emptyset.$$

Otherwise there will be no restriction on the choice of direct sums.

Given two maps $A_1 \to B_1$, $A_2 \to B_2$, we shall write $A_1B_1 \oplus A_2B_2$ for the map $A_1 \oplus A_2 \to B_1 \oplus B_2$ given by $(A_1 \oplus A_2)\mathfrak{d}A_1B_1c(B_1 \oplus B_2) + (A_1 \oplus A_2)\mathfrak{d}A_2B_2c(B_1 \oplus B_2)$. Direct summing of maps preserves kernels, cokernels, and therefore exactness of sequences. For two classes $\mathcal{S}_1$, $\mathcal{S}_2$ of maps in an additive category, we shall denote by $\mathcal{S}_1 \oplus \mathcal{S}_2$ the totality of maps which can be presented as the direct sum $\varphi_1 \oplus \varphi_2$ ($\varphi_1 \in \mathcal{S}_1$ $\varphi_2 \in \mathcal{S}_2$). Clearly we have $\mathrm{P}\mathcal{A} \oplus \mathrm{P}\mathcal{A} = \mathrm{P}\mathcal{A}$. If $\mathcal{A}$ is additive then the categories $\mathcal{E}^n\mathcal{A}$ $(n = 0, 1, 2, \cdots)$ are additive. Given a map $\varphi: A \to B$ we define $c^\varphi: A \to A \oplus B$, $\mathfrak{d}^\varphi: A \oplus B \to B$, $c_\varphi: A \to B \oplus A$, $\mathfrak{d}_\varphi: B \oplus A \to B$ by

$$Ac^\varphi(A \oplus B) = Ac(A \oplus B) + ABc(A \oplus B),$$
$$(A \oplus B)\mathfrak{d}^\varphi B = -(A \oplus B)\mathfrak{d}AB + (A \oplus B)\mathfrak{d}B,$$
$$Ac_\varphi(B \oplus A) = -ABc(B \oplus A) + Ac(B \oplus A),$$
$$(B \oplus A)\mathfrak{d}_\varphi B = (B \oplus A)\mathfrak{d}B + (B \oplus A)\mathfrak{d}AB.$$

Then we have

$$(Ac^\varphi(A \oplus B), Bc(A \oplus B), (A \oplus B)\mathfrak{d}A, (A \oplus B)\mathfrak{d}^\varphi B): \ A \oplus B \approx A \oplus B,$$
$$(Bc(B \oplus A), Ac_\varphi(B \oplus A), (B \oplus A)\mathfrak{d}_\varphi B, (B \oplus A)\mathfrak{d}A): \ B \oplus A \approx B \oplus A,$$

and $Ac^\varphi(A \oplus B)\mathfrak{d}B = Ac(B \oplus A)\mathfrak{d}_\varphi B = AB$.

In the special case $\varphi=e_A$, the maps c^{φ}, $\mathfrak{d}_{\varphi}$ reduce to the diagonal map $\Delta_A: A \to A\oplus A$ and the codiagonal map $\nabla_A: A\oplus A\to A$.

A map $A\to B$ is called *direct* if it is proper and if there is a map $B\to A$ such that $ABAB=AB$. The totality of direct maps (direct injections, direct projections) will be denoted by $\mathrm{D}\mathcal{A}$ ($\mathrm{DI}\mathcal{A}$, $\mathrm{DJ}\mathcal{A}$). Clearly we have

$$\mathrm{D}\mathcal{A}\supset\mathrm{E}\mathcal{A}$$
$$\mathrm{D}\mathcal{A}\oplus\mathrm{D}\mathcal{A}=\mathrm{D}\mathcal{A}, \qquad \mathrm{DI}\mathcal{A}\circ\mathrm{DI}\mathcal{A}=\mathrm{DI}\mathcal{A},$$
$$\mathrm{DJ}\mathcal{A}\circ\mathrm{DJ}\mathcal{A}=\mathrm{DJ}\mathcal{A}, \qquad \mathrm{DI}\mathcal{A}\circ\mathrm{DJ}\mathcal{A}=\mathrm{D}\mathcal{A}.$$

The four maps in the direct sum $A\approx A_1\oplus A_2$ are all direct. Conversely, in an exact sequence

$$X:\ \emptyset\to B\to E\to A\to\emptyset,$$

suppose EA is direct. Then we have a map $A\to E$ such that $AEA=AeA$. Since $(EeE-EAE)\cdot EA=0$, there is a unique map $E\to B$ such that $EBE=EeE-EAE$. From $BEBE=BE-BEAE=BE$ follows $BEB=BeB$, and so we get

$$(AE, BE, EA, EB):\ E\approx A\oplus B.$$

In this case the short exact sequence X is said to be *direct*. Also BE is direct if and only if X is direct. From this we see easily the identity.

$$\mathrm{kernel}^{-1}\mathrm{D}\mathcal{A}\cap\mathrm{cokernel}^{-1}\,\mathrm{D}\mathcal{A}=\mathrm{D}\mathcal{A}.$$

Note that we have $\mathrm{DJ}\mathcal{A}\circ\mathrm{DI}\mathcal{A}=\mathcal{A}$, for any map $\varphi: A\to B$ can be decomposed as $AB=Ac^{\varphi}(A\oplus B)\mathfrak{d}B$.

1.4. In what follows the commutative diagrams

(3×2)
$$\begin{array}{lccccccccc} \boldsymbol{A}: & \emptyset & \to & \dot{A} & \to & A & \to & \bar{A} & \to & \emptyset \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ \boldsymbol{B}: & \emptyset & \to & \dot{B} & \to & B & \to & \bar{B} & \to & \emptyset, \end{array}$$

(3×2′)
$$\begin{array}{lccccccccc} \boldsymbol{A}': & \emptyset & \to & \dot{B} & \to & A' & \to & \bar{A}' & \to & \emptyset \\ & & & \Downarrow & & \downarrow & & \downarrow & & \\ \boldsymbol{B}: & \emptyset & \to & \dot{B} & \to & B & \to & \bar{B} & \to & \emptyset \end{array}$$

with the sequences $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{A}'$ exact or not, will appear frequently. These will be referred to as the (3×2)-diagram, (3×2′)-diagram respectively.

LEMMA 3×2. 1. *In the* (3×2)-*diagram suppose* $\dot{A}A=\mathrm{kernel}\,A\bar{A}$ *and* $\dot{B}B$, $\bar{A}\bar{B}$ *are injections.*

(i) *If* $\dot{B}B\bar{B}=0$ *and if* AB *is the kernel of a map* $B\to C$, *then* $\dot{A}\dot{B}=\mathrm{kernel}\,\dot{B}BC$.

(ii) *If* $\dot{A}\dot{B}$ *has the kernel* $\dot{N}\to\dot{A}$, *then* $\dot{N}\dot{A}A=\mathrm{kernel}\,AB$.

(iii) *If* AB *has the kernel* $N\to A$, *then there is a unique map* $N\to\dot{A}$ *with* $N\dot{A}A=NA$, *and we have* $N\dot{A}=\mathrm{kernel}\,\dot{A}\dot{B}$.

(iv) *$\dot{A}\dot{B}$ is an injection if and only if AB is an injection.*

PROOF. Ad (i): Firstly we have $\dot{A}\dot{B}BC=\dot{A}ABC=0$. Secondly $\dot{A}\dot{B}$ is an injection, since $\dot{A}\dot{B}B=\dot{A}AB$ is an injection. Let $D\to\dot{B}$ be any map with $D\dot{B}BC=0$. Then there is a map $D\to A$ such that $DAB=D\dot{B}B$. From $DA\bar{A}\bar{B}=DAB\bar{B}=D\dot{B}B\bar{B}=0$ follows $DA\bar{A}=0$, and so there is a map $D\to\dot{A}$ such that $D\dot{A}A=DA$. This map gives $D\dot{A}\dot{B}=D\dot{B}$, for we have $D\dot{A}\dot{B}B=D\dot{A}AB=DAB=D\dot{B}B$. So we get $\dot{A}\dot{B}=\text{kernel}\,\dot{B}BC$.

Ad (ii): Firstly we have $\dot{N}\dot{A}AB=\dot{N}\dot{A}\dot{B}B=0$. Secondly $\dot{N}\dot{A}A$ is an injection, because $\dot{N}\dot{A}$, $\dot{A}A$ are injections. Let $D\to A$ be any map with $DAB=0$. From $DA\bar{A}\bar{B}=DAB\bar{B}=0$ follows $DA\bar{A}=0$, and so there is a map $D\to\dot{A}$ such that $D\dot{A}A=DA$. Next $D\dot{A}\dot{B}B=D\dot{A}AB=DAB=0$ implies $D\dot{A}\dot{B}=0$, and so there is a map $D\to\dot{N}$ such that $D\dot{N}\dot{A}=D\dot{A}$. This map gives $D\dot{N}\dot{A}A=D\dot{A}A=DA$, which proves $\dot{N}\dot{A}A=\text{kernel}\,AB$.

Ad (iii): Because of $NA\bar{A}\bar{B}=NAB\bar{B}=0$ we get $NA\bar{A}=0$, and so there is a unique map $N\to\dot{A}$ such that $N\dot{A}A=NA$. Applying the part (i) to the commutative diagram

$$\begin{array}{ccccc} N & \Rightarrow & N & \to & \emptyset \\ \downarrow & & \downarrow & & \downarrow \\ \dot{A} & \to & A & \to & \bar{A} \end{array}$$

we obtain $N\dot{A}=\text{kernel}\,\dot{A}AB$. Since $\dot{B}B$ is an injection we get $\text{kernel}\,\dot{A}AB=\text{kernel}\,\dot{A}\dot{B}B=\text{kernel}\,\dot{A}\dot{B}$, and so $N\dot{A}=\text{kernel}\,\dot{A}\dot{B}$.

Ad (iv): The 'if' part is obvious, while the 'only if' part is the special case $\dot{N}=\emptyset$ of (ii). This completes the proof.

LEMMA 3×2. 2. *In the* $(3\times2')$-*diagram suppose A' is exact and $\dot{B}B=\text{kernel}$ $B\bar{B}$. Asumme there is a map $\bar{A}'\to B$ with $\bar{A}'B\bar{B}=\bar{A}'\bar{B}$. Then there exist maps $A'\to\dot{B}$, $\bar{A}'\to A'$ such that $A'\dot{B}B=A'B-A'\bar{A}'B$, $(\bar{A}'A', \dot{B}A', A'\bar{A}', A'\dot{B})$: $A'\approx\bar{A}'\oplus\dot{B}$, and $\bar{A}'A'B=\bar{A}'B$.*

PROOF. Becauee of $(A'B-A'\bar{A}'B)\cdot B\bar{B}=A'B\bar{B}-A'\bar{A}'\bar{B}=0$, there is a map $A'\to\dot{B}$ such that $A'\dot{B}B=A'B-A'\bar{A}'B$. Further from $\dot{B}A'\dot{B}B=\dot{B}A'B-\dot{B}A'\bar{A}'B=\dot{B}A'B-0=\dot{B}B$ follows $\dot{B}A'\dot{B}=\dot{B}e\dot{B}$. Thus $\dot{B}A'$ being direct, the short exact sequence A' is direct, and so we have a map $\bar{A}'\to A'$ completing ths direct sum $(\bar{A}'A'$, $\dot{B}A', A'\bar{A}', A'\dot{B})$: $A'\approx\bar{A}'\oplus\dot{B}$. Finally we have $\bar{A}'A'B=\bar{A}'A'\cdot(A'\dot{B}B+A'\bar{A}'B)=\bar{A}'A'\dot{B}B+\bar{A}'A'\bar{A}'B=0+\bar{A}'A'\bar{A}'B=\bar{A}'B$, which completes the proof.

LEMMA 3×2. 2. *In the* $(3\times2')$-*diagram suppose A' is direct exact, and $\dot{B}B=\text{kernel}\,B\bar{B}$. Then $A'B$ is an injection if and only if $\bar{A}'\bar{B}$ is an injection.*

PROOF. The 'if' part is obvionus by Lemma 3×2. 1, (iv). For the 'only if' part let $C\to\bar{A}'$ be any map with $C\bar{A}'\bar{B}=0$. By assumption we have a map $\bar{A}'\to A'$

such that $\bar{A}'A'\bar{A}'=\bar{A}'e\bar{A}'$. Thus because of $C\bar{A}'A'B\cdot B\bar{B}=C\bar{A}'A'\bar{A}'\bar{B}'=C\bar{A}'\bar{B}'=0$ there is a map $C\to\dot{B}$ with $C\dot{B}B=C\bar{A}'A'B$. If now $A'B$ is an injection, then from $C\bar{A}'A'B=C\dot{B}B=C\dot{B}A'B$ follows $C\bar{A}'A'=C\dot{B}A'$, and so we get $C\bar{A}'=C\bar{A}'A'\bar{A}'=C\dot{B}A'\bar{A}'=0$, completing the proof.

1.5. In the (3×2)-diagram suppose A, B are exact and $\dot{A}\dot{B}$, AB, $\bar{A}\bar{B}$ are proper projections with respective kernels $\dot{N}\to\dot{A}$, $N\to A$, $\bar{N}\to\bar{A}$. Then we get the commutative diagram

$$\begin{array}{ccccccccc}
 & & \emptyset & & \emptyset & & \emptyset & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
N: & \emptyset\to & \dot{N} & \to & N & \to & \bar{N} & \to\emptyset & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
A: & \emptyset\to & \dot{A} & \to & A & \to & \bar{A} & \to\emptyset & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
B: & \emptyset\to & \dot{B} & \to & B & \to & \bar{B} & \to\emptyset & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \emptyset & & \emptyset & & \emptyset & &
\end{array}$$

where rows and columns are exact except for the top row N. By Lemma 3×2. 1, (i) we have $\dot{N}N=\text{kernel}\,NA\bar{A}=\text{kernel}\,N\bar{N}\bar{A}=\text{kernel}\,N\bar{N}$. As stated in [1] we have:

LEMMA 3×3. 1. *N is exact if $\mathcal{A}$ is an exact category.*

Perhaps the shortest proof is as follows. Firstly N is exact if $N\bar{N}$ is a projection. For in an exact category a projection is always a proper projection, and so it is the cokernel of its kernel. By duality, if in the above diagram rows and columns are exact except for the third column, and if $\bar{N}\bar{A}$ is an injection, then the third column is also exact. Secondly $N\bar{N}$ being proper, we take $(N\to M)=\text{coimage}\,N\bar{N}=\text{cokernel}\,\dot{N}N$ and $(M\to\bar{N})=\text{image}\,N\bar{N}$. Define $M\to\bar{A}$ by $M\bar{A}=M\bar{N}\bar{A}$. Then $M\bar{A}$ is an injection, and so by the above result the sequence $\emptyset\to M\to\bar{A}\to\bar{B}\to\emptyset$ is exact. Therefore $M\bar{N}$ must be an equivalence map. Hence N is exact.

It is clear from Lemma 3×3.1 that in the category $\mathcal{E}^n\mathcal{A}$ of n-fold extensions in an exact $\mathcal{A}$ a map $X\to X'$ is a proper projection (proper injection) if all the vertical constituents are projections (injections). It is not difficult to see that in $\mathcal{E}^1\mathcal{A}$ the converse holds. Note that the converse does not hold in $\mathcal{E}^2\mathcal{A}$, as illustrated by the following short exact sequence in $\mathcal{E}^2\mathcal{A}$:

$$\begin{array}{ccccccccccc}
\emptyset & \to & \emptyset & \to & \emptyset & \to & C & \Rightarrow & C & \to & \emptyset \\
 & & \downarrow & & \downarrow & & \Downarrow & & \downarrow & & \\
\emptyset & \to & \emptyset & \to & C & \Rightarrow & C & \to & \emptyset & \to & \emptyset \\
 & & \downarrow & & \Downarrow & & \downarrow & & \downarrow & & \\
\emptyset & \to & C & \Rightarrow & C & \to & \emptyset & \to & \emptyset & \to & \emptyset.
\end{array}$$

Suppose $\mathcal{A}$ is an exact category. Let X: $\emptyset\to B\to E\to A\to\emptyset$ be an exact sequence, and let $\dot{A}\to A$ be a direct injection with the cokernel $A\to\bar{A}$. Then

taking the kernel $\dot{E}\to E$ of the projection $EA\bar{A}$, we get by Lemma 3×3.1 (or by Lemma 3×2.1, (iii) and Lemma 3×2.3) the commutative diagram

$$\begin{array}{ccccccccccc} X': & \emptyset & \to & B & \to & \dot{E} & \to & \dot{A} & \to & \emptyset \\ & & & \Downarrow & & \downarrow & & \downarrow & & \\ X: & \emptyset & \to & B & \to & E & \to & A & \to & \emptyset \end{array}$$

with exact rows. We now introduce the following conditions to be imposed on an additive category.

(Q1) Any short exact sequence $X\colon \emptyset\to B\to E\to A\to\emptyset$ and any direct injection $\dot{A}\to A$ can be extended to a map $\dot{X}\to X$ in $\mathcal{E}^1\mathcal{A}$ such that $E_+^1(\dot{X}X)=BeB$.

(Q1*) Any short exact sequence $X\colon \emptyset\to B\to E\to A\to\emptyset$ and any direct projection $B\to\bar{B}$ can be extended to a map $X\to\bar{X}$ in $\mathcal{E}^1\mathcal{A}$ such that $E_-^1(X\bar{X})=AeA$.

(Q2) Any short exact sequence $X\colon \emptyset\to B\to E\to A\to\emptyset$ and any map $A'\to A$ can be extended to a map $X'\to X$ in $\mathcal{E}^1\mathcal{A}$ such that $E_+^1(X'X)=BeB$.

(Q2*) Any short exact sequence $X\colon \emptyset\to B\to E\to A\to\emptyset$ and any map $B\to B'$ can be extended to a map $X\to X'$ in $\mathcal{E}^1\mathcal{A}$ such that $E_-^1(XX')=AeA$.

Obviously (Q2) implies (Q1). We assert the converse. In fact given X and $\alpha\colon A'\to A$ we take the direct sum sequence $\tilde{X}\colon \emptyset\to B\to A'\oplus E\to A'\oplus A\to\emptyset$ of X and $\emptyset\to\emptyset\to A'\Rightarrow A'\to\emptyset$. Since $A'c^{\alpha}(A'\oplus A)$ is a direct injection we have by (Q1) a map $X'\to\tilde{X}$ in $\mathcal{E}^1\mathcal{A}$ extending $A'c^{\alpha}(A'\oplus A)$ and BeB. This composed with the canonical projection $\tilde{X}\to X$ gives the required map in (Q2). Dually (Q2*) is equivalent to (Q1*).

An exact category satisfies (Q1), (Q1*), and therefore an abelian category satisfies (Q2), (Q2*). However, (Q2) and (Q2*) are not exclusive properties of an abelian category. In fact if $\mathcal{A}$ is abelian, then the non-abelian category $\mathcal{E}^1\mathcal{A}$ satisfies (Q1), (Q1*), and so (Q2), (Q2*). Also the additive category $\mathcal{C}_{\mathrm{A}}$ satisfies (Q2), (Q2*) (cf. [9, VIII].)

In proving Lemma 3×3.1 in the category ${}_{\Lambda}\mathfrak{M}$ of Λ-modules, one would usually proceed by picking up elements of Λ-modules and by checking whether a certain element of a Λ-module lies in the image of a certain Λ-homomorphism. For any map $A'\to A$ and any epimorphism $E\to A$ in ${}_{\Lambda}\mathfrak{M}$, the image in A of an element of A' can be 'lifted' to an element of E. This type of 'elementary' argument is not allowed in an abstract additive category. In ${}_{\Lambda}\mathfrak{M}$ we can avoid such an elementary argument in covering A' by an epimorphism $F'\to A'$ from a Λ-free module F', whence $A'\to A$ can be lifted to a map $F'\to E$. In an exact category with sufficiently many projectives a projection $F'\to A'$ from a projective object P' will do the work. The use of projectives can also be avoided in an additive category satisfying the following conditions:

(Q2.1) Any short exact sequence $X\colon \emptyset\to B\to E\to A\to\emptyset$ and any map $A'\to A$ can be extended to a map $X'\to X$ in $\mathcal{E}^1\mathcal{A}$.

(Q2.1*) Any short exact sequence X: $\emptyset \to B \to E \to A \to \emptyset$ and any map $B \to B'$ can be extended to a map $X \to X'$ in $\mathcal{E}^1\mathcal{A}$.

These are weakened forms of (Q2), (Q2*) respectively, where again to the same effect we may restrict $A' \to A$ to direct injections and $B \to B'$ to direct projections. Note that (Q2.1) ((Q2.1*)) holds in an additive category with sufficiently many projectives (injectives). We shall illustrate the use of (Q2.1) in proving the following generalization of Lemma 3×3.1*, namely:

LEMMA 3×3.2. *Assume $\mathcal{A}$ satisfies* (Q2.1). *In the commutative diagram*

$$\begin{array}{ccccccccc}
 & & \dot{X}: & & X: & & \bar{X}: & & \\
 & & \emptyset & & \emptyset & & \emptyset & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
B: & \emptyset \to & \dot{B} & \to & B & \to & \bar{B} & \to \emptyset & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
E: & \emptyset \to & \dot{E} & \to & E & \to & \bar{E} & \to \emptyset & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
A: & \emptyset \to & \dot{A} & \to & A & \to & \bar{A} & \to \emptyset & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \emptyset & & \emptyset & & \emptyset & &
\end{array}$$

suppose rows and columns are exact except for the bottom row A. *Then* A *is necessarily exact.*

PROOF. We know $A\bar{A}$=cokernel $\dot{A}A$ by Lemma 3×2.1, (i)*. So we shall prove $\dot{A}A$=kernel $A\bar{A}$. Let $C \to \dot{A}$ be any map with $C\dot{A}A=0$. Extend it to a commutative diagram

$$\begin{array}{cccccccc}
Y: & \emptyset \to & D & \to & F & \to & C & \to \emptyset \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\dot{X}: & \emptyset \to & \dot{B} & \to & \dot{E} & \to & \dot{A} & \to \emptyset
\end{array}$$

with exact rows. Because of $F\dot{E}EA=F\dot{E}\dot{A}A=FC\dot{A}A=0$, there is a map $F \to B$ such that $FBE=F\dot{E}E$. Then from $FB\bar{B}\bar{E}=FBE\bar{E}=F\dot{E}E\bar{E}=0$ follows $FB\bar{B}=0$, and so there is a map $F \to \dot{B}$ such that $F\dot{B}B=FB$. Further from $F\dot{B}\dot{E}E=F\dot{B}BE$ $=FBE=F\dot{E}E$ follows $F\dot{B}\dot{E}=F\dot{E}$. Therefore we get $FC\dot{A}=F\dot{E}\dot{A}=F\dot{B}\dot{E}\dot{A}=0$, and so $C\dot{A}=0$. Thus $\dot{A}A$ is an injection. Next let $\bar{C} \to A$ be any map with $\bar{C}A\bar{A}$ $=0$. Again extend it to a commutative diagram

$$\begin{array}{cccccccc}
\bar{Y}: & \emptyset \to & \bar{D} & \to & \bar{F} & \to & \bar{C} & \to \emptyset \\
 & & \downarrow & & \downarrow & & \downarrow & \\
X: & \emptyset \to & B & \to & E & \to & A & \to \emptyset
\end{array}$$

with exact rows. Because of $\bar{F}E\bar{E}\bar{A}=\bar{F}EA\bar{A}=\bar{F}\bar{C}A\bar{A}=0$, there is a map $\bar{F} \to \bar{B}$ such that $\bar{F}\bar{B}\bar{E}=\bar{F}E\bar{E}$. Extend this map to a commutative diagram

$$\begin{array}{cccccccc}
F: & \emptyset \to & \dot{F} & \to & F & \to & \bar{F} & \to \emptyset \\
 & & \downarrow & & \downarrow & & \downarrow & \\
B: & \emptyset \to & \dot{B} & \to & B & \to & \bar{B} & \to \emptyset
\end{array}$$

with exact rows. Since we have $(F\bar{F}E-FBE)\cdot E\bar{E}=F\bar{F}\bar{B}\bar{E}-FB\bar{B}\bar{E}=0$, there is a map $F\to\dot{E}$ such that $F\dot{E}E=F\bar{F}E-FBE$. Then from $\dot{F}F\dot{E}\dot{A}A=\dot{F}F\dot{E}EA$ $=\dot{F}F\bar{F}EA-\dot{F}FBEA=0$ follows $\dot{F}F\dot{E}\dot{A}=0$, for $\dot{A}A$ is already shown to be an injection. Consequently there is a map $\dot{F}\to\dot{A}$ such that $F\dot{F}\dot{A}=F\dot{E}\dot{A}$. Finally extend the map $\bar{D}\bar{F}$ to a commutative diagram

$$\begin{array}{rccccccccc}
D: & \emptyset & \to & \dot{D} & \to & D & \to & \bar{D} & \to & \emptyset \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
F: & \emptyset & \to & \dot{F} & \to & F & \to & \bar{F} & \to & \emptyset
\end{array}$$

with exact rows. Then we have $D\bar{D}\bar{F}\dot{A}A=DF\bar{F}\dot{A}A=DF\dot{E}\dot{A}A=DF\dot{E}EA$ $=DF\bar{F}EA-DFBEA=DF\bar{F}EA=D\bar{D}\bar{F}\bar{C}A=0$, and so $\bar{D}\bar{F}\dot{A}=0$. Hence there is a map $\bar{C}\to\dot{A}$ such that $\bar{F}\bar{C}\dot{A}=\bar{F}\dot{A}$. For this map we have $\bar{C}\dot{A}A=\bar{C}A$ because of $F\bar{F}\bar{C}\dot{A}A=F\bar{F}\dot{A}A=F\dot{E}\dot{A}A=F\dot{E}EA=F\bar{F}EA-FBEA=F\bar{F}EA=F\bar{F}\bar{C}A$. Thus $\dot{A}A$ $=\text{kernel}\, A\bar{A}$, and the proof is completed.

§2. S-categories.

2.0. In §1 we met with examples of non-abelian additive categories. Let $\mathcal{A}$ stand for one of the categories $\mathcal{E}^n\mathcal{A}^0$ ($n\geqq 1$, $\mathcal{A}^0$ is abelian), $\mathcal{C}_{\mathrm{A}}$. We shall say for the time being that a short exact sequence $\emptyset\to B\to E\to A\to\emptyset$ in $\mathcal{A}$ is 'special' if for $\mathcal{A}=\mathcal{E}^n\mathcal{A}^0$ the $n+2$ constituents $\emptyset\to E_k^n(B)\to E_k^n(E)\to E_k^n(A)\to\emptyset$ ($k=-,1,2,\cdots,n,+$) in $\mathcal{A}^0$ are all exact, and if for $\mathcal{A}=\mathcal{C}_{\mathrm{A}}$ the sequence is 'strictly exact' (cf. [9, VII]). We shall also say that a proper injection $B\to E$ in $\mathcal{A}$ is special if the short exact sequence $\emptyset\to B\to E\to \operatorname{Coker} BE\to\emptyset$ is special, and that a proper projection $E\to A$ in $\mathcal{A}$ is special if the short exact sequence $\emptyset\to \operatorname{Ker} EA\to E\to A\to\emptyset$ is special. Then a common feature of $\mathcal{A}=\mathcal{E}^n\mathcal{A}$, and $\mathcal{A}=\mathcal{C}_{\mathrm{A}}$ is that in $\mathcal{A}$ the following conditions hold:

(Q0) (special projection)$\circ$(special$\circ$projection)$=$(special projection).

(Q0*) (special injection)$\circ$(special injection)$=$(special injection).

(Q2) Any special exact sequence $X:\emptyset\to B\to E\to A\to\emptyset$ and any map $A'\to A$ can be imbedded in a commutative diagram

$$\begin{array}{rccccccccc}
X': & \emptyset & \to & B & \to & E' & \to & A' & \to & \emptyset \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
X: & \emptyset & \to & B & \to & E & \to & A & \to & \emptyset
\end{array}$$

with special exact rows.

(Q2*) Any special exact sequence $X:\emptyset\to B\to E\to A\to\emptyset$ and any map $B\to B'$ can be imbedded in a commutative diagram

$$\begin{array}{rccccccccc}
X: & \emptyset & \to & B & \to & E & \to & A & \to & \emptyset \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
X': & \emptyset & \to & B' & \to & E' & \to & A & \to & \emptyset
\end{array}$$

with special exact rows.

In generalizing those additive categories we introduce the following:

DEFINITION. *An* S-*category is a pair* $(\mathcal{A}, \mathcal{S})$ *of an addive category* $\mathcal{A}$ *and a class* $\mathcal{S}$ *of maps in* $\mathcal{A}$, *subject to the following four conditions:*

(S1) $\mathrm{P}\mathcal{A} \supset \mathcal{S} \cap \mathrm{E}\mathcal{A}$.

(S2) $\mathcal{S} \oplus \mathcal{S} \subset \mathcal{S}$.

(S3) kernel $\mathcal{S} \subset \mathcal{S}$, cokernel $\mathcal{S} \subset \mathcal{S}$.

(S4) $\mathcal{S}\mathrm{I} \circ \mathcal{S}\mathrm{J} = \mathcal{S}$ (*where* $\mathcal{S}\mathrm{I} = \mathcal{S} \cap \mathrm{I}\mathcal{A}$, $\mathcal{S}\mathrm{J} = \mathcal{S} \cap \mathrm{J}\mathcal{A}$).

Maps belonging to $\mathcal{S}$ *will be called special* (*or* $\mathcal{S}$-*special*) *maps. An exact sequence in* $\mathcal{A}$ *is called special* (*or* $\mathcal{S}$-*special*) *if every map in the sequence is special. A regular* S-*category is an* S-*category satisfying* (Q2), (Q2*). *A quasi-abelian* S-*category is an* S-*category satisfying* (Q0), (Q0*), (Q2), (Q2*).

The conditions (S3), (S4) can be unified to the condition that in an exact sequence $A \to B \to C \to D$ if AB, CD are special, then BC is also special. Among S-categories attached to an additive category $\mathcal{A}$, the pair $(\mathcal{A}, \mathrm{D}\mathcal{A})$ is the smallest, and $(\mathcal{A}, \mathrm{P}\mathcal{A})$ is the largest. For any number of S-categories $(\mathcal{A}, \mathcal{S}_j)$ the intersection $(\mathcal{A}, \cap \mathcal{S}_j)$ is again an S-category. Clearly $(\mathcal{A}, \mathrm{D}\mathcal{A})$ is quasi-abelian. An additive category $\mathcal{A}$ is called *quasi-abelian* if $(\mathcal{A}, \mathrm{P}\mathcal{A})$ is regular. In this case $(\mathcal{A}, \mathrm{P}\mathcal{A})$ is quasi-abelian, as will be shown later. The author does not know whether a regular S-category is always quasi-abelian. Note that the argument in §1.5 shows that (Q2) is equivalent to its weakened form (Q1) where we restrict $A' \to A$ to be a direct injection. Dually (Q2*) is equivalent to its weakened form (Q1*) where we restrict $B \to B'$ to be a direct projection. For an abelian category $\mathcal{A}$ the S-category $(\mathcal{A}, \mathrm{P}\mathcal{A})$ satisfies (Q0), (Q0*), while (Q2), (Q2*) have been verified in §1.5. Thus an abelian category is always quasi-abelian.

2.1. We prepare some lemmas on the (3×2)-diagram

$$\begin{array}{rccccccccc} A: & \emptyset & \to & \dot{A} & \to & A & \to & \bar{A} & \to & \emptyset \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ B: & \emptyset & \to & \dot{B} & \to & B & \to & \bar{B} & \to & \emptyset, \end{array}$$

and the $(3\times 2')$-diagram

$$\begin{array}{rccccccccc} A': & \emptyset & \to & \dot{B} & \to & A' & \to & \bar{A}' & \to & \emptyset \\ & & & \Downarrow & & \downarrow & & \downarrow & & \\ B: & \emptyset & \to & \dot{B} & \to & B & \to & \bar{B} & \to & \emptyset. \end{array}$$

LEMMA 3×2.4. *Assume* (Q2). *In the* $(3\times 2')$-*diagram suppose* A' *is special exact and* $\dot{B}B = \mathrm{kernel}\, B\bar{B}$.

(i) $\bar{A}'\bar{B}$ *is an injection if and only if* $A'B$ *is an injection.*

(ii) *If* $\bar{A}'\bar{B}$ *is the kernel of a map* $\bar{B} \to C$, *then* $A'B = \mathrm{kernel}\, B\bar{B}C$.

(iii) *If* $\bar{A}'\bar{B}$ *is an equivalence map, then* $A'B$ *is an equivalence map, and*

B is special exact.

PROOF. Ad (i): The 'only if' part is in cluded in Lemma 3×2.1, (iv). For the 'if' part let $\bar{D}\to\bar{A}'$ be any map with $\bar{D}\bar{A}'\bar{B}=0$. In virtue of (Q2) we can imbed A' and $\bar{D}\bar{A}'$ in a commutative diagram

$$\begin{array}{cccccccccc} D: & \emptyset & \to & \dot{B} & \to & D & \to & \bar{D} & \to & \emptyset \\ & & & \Downarrow & & \downarrow & & \downarrow & & \\ A': & \emptyset & \to & \dot{B} & \to & A' & \to & \bar{A}' & \to & \emptyset \end{array}$$

with special exact rows. Because of $DA'B\bar{B}=DA'\bar{A}'\bar{B}=D\bar{D}\bar{A}'\bar{B}=0$ there is a map $D\to\dot{B}$ such that $D\dot{B}B=DA'B$. Since $A'B$ is assumed to be an injection, from $D\dot{B}A'B=D\dot{B}B=DA'B$ follows $D\dot{B}A'=DA'$. Therefore we get $D\bar{D}\bar{A}'=DA'\bar{A}'=D\dot{B}A'\bar{A}'=0$, and so $\bar{D}\bar{A}'=0$. Thus $\bar{A}'\bar{B}$ is an injection.

Ad (ii): Firstly we have $A'B\bar{B}C=A'\bar{A}'\bar{B}C=0$. Secondly $A'B$ in an injection, since $\bar{A}'\bar{B}=\text{kernel}\,\bar{B}C$ is an injection. Now let $\bar{D}\to B$ be any map with $\bar{D}B\bar{B}C=0$. Then there is a map $\bar{D}\to\bar{A}'$ such that $\bar{D}\bar{A}'\bar{B}=\bar{D}B\bar{B}$. Extend A' and $\bar{D}\bar{A}'$ to a commutative diagram $D\to A'$ as above. By composing the vertical constituents in $D\to A'$ with those in the $(3\times 2')$-diagram we obtain the commutative diagram

$$\begin{array}{cccccccccc} D: & \emptyset & \to & \dot{B} & \to & D & \to & \bar{D} & \to & \emptyset \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ B: & \emptyset & \to & \dot{B} & \to & B & \to & \bar{B} & \to & \emptyset. \end{array}$$

Since we have $\bar{D}B\bar{B}=\bar{D}\bar{A}'\bar{B}=\bar{D}\bar{B}$, Lemma 3×2.2 can now be applied, and we obtain a map $\bar{D}\to D$ such that $\bar{D}DB=\bar{D}B$. So the map $\bar{D}\to A'$ defined by $\bar{D}A'=\bar{D}DA'$ gives $\bar{D}A'B=\bar{D}DA'B=\bar{D}DB=\bar{D}B$. This shows $A'B=\text{kernel}\,B\bar{B}C$.

Ad (iii): We now have $\bar{A}'\bar{B}=\text{kernel}\,\bar{B}\emptyset$, and so by the above part (ii) we get $A'B=\text{kernel}\,B\emptyset$. Thus $A'B$ is an equivalence map, and so, being at all equivalent to the special exact A', the sequence B is special exact.

LEMMA 3×2.5. *Assume* (Q2). *In the* $(3\times 2')$*-diagram suppose B is special exact. Suppose further that $A'B$ is the kernel of a map $B\to C$. Then there is a unique map $\bar{B}\to C$ such that $B\bar{B}C=BC$. Moreover A' is special exact if and only if $\bar{A}'\bar{B}=\text{kernel}\,\bar{B}C$.*

PROOF. The first part is obvious from $\dot{B}BC=\dot{B}A'BC=0$. Suppose $\bar{A}'\bar{B}'=\text{kernel}\,\bar{B}C$. Since this is an injection we get $\dot{B}A'=\text{kernel}\,A'\bar{A}'$ by Lemma 3×2.1, (iii). Now in virtue of (Q2) we can imbed B and $\bar{A}'\bar{B}$ in a commutative diagram

$$\begin{array}{cccccccccc} A_1: & \emptyset & \to & \dot{B} & \to & A_1 & \to & \bar{A}' & \to & \emptyset \\ & & & \Downarrow & & \downarrow & & \downarrow & & \\ B: & \emptyset & \to & \dot{B} & \to & B & \to & \bar{B} & \to & \emptyset \end{array}$$

with special exact rows. Because of $A_1B\bar{B}C=A_1\bar{A}'\bar{B}C=0$ there is a map $A_1\to A'$ such that $A_1A'B=A_1B$. Then from $A_1A'\bar{A}'\bar{B}=A_1A'B\bar{B}=A_1B\bar{B}=A_1\bar{A}'\bar{B}$ follows

$A_1A'\bar{A}'=A_1\bar{A}'$, and from $\dot{B}A_1A'B=\dot{B}A_1B=\dot{B}A'B$ follows $\dot{B}A_1A'=\dot{B}A'$. Thus commutativity holds in the diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
A_1\colon & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d, Rightarrow] & A_1 \arrow[r] \arrow[d] & \bar{A} \arrow[r] \arrow[d] & \emptyset \\
A'\colon & \emptyset \arrow[r] & \dot{B} \arrow[r] & A' \arrow[r] & \bar{A}' \arrow[r] & \emptyset
\end{tikzcd}$$

and so by Lemma 3×2.4, (iii) the sequence A' is special exact.

LEMMA 3×2.6. *Assume* (Q2). *In the* (3×2)-*diagram and* $(3\times2')$-*diagram with the same* $\boldsymbol{B}$ *suppose* $\dot{A}A\bar{A}=0$, $\dot{B}B=\mathrm{kernel}\,B\bar{B}$, *and* A' *is special exact. Further assume that* $\bar{A}\bar{B}$ *is decomposed as* $\bar{A}\bar{B}=\bar{A}\bar{A}'\bar{B}$ *by a map* $\bar{A}\to\bar{A}'$. *Then there is a unique map* $A\to A'$ *such that commutativity holds in the diagram*

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \dot{A} \arrow[rr] \arrow[dr] \arrow[dd] & & A \arrow[rr] \arrow[dr] \arrow[dd] & & \bar{A} \arrow[dr] \arrow[dd] & & \\
\emptyset \arrow[rr] & & \dot{B} \arrow[dl, Rightarrow] \arrow[rr] & & A' \arrow[dl] \arrow[rr] & & \bar{A}' \arrow[dl] \arrow[r] & \emptyset \\
 & \dot{B} \arrow[rr] & & B \arrow[rr] & & \bar{B} & &
\end{tikzcd}$$

PROOF. Consider the direct sum sequence $\dot{B}\to\bar{A}'\oplus B\to\bar{A}'\oplus\bar{B}$ of $\emptyset\to\bar{A}'\Rightarrow\bar{A}'$ and $\dot{B}\to B\to\bar{B}$. We have $\dot{B}(\bar{A}'\oplus B)=\mathrm{kernel}\,(\bar{A}'\oplus B)(\bar{A}'\oplus\bar{B})$. Naming the map $A'B$ as φ, we have the direct exact sequence $\emptyset\to\bar{A}'\xrightarrow{c^{\varphi}}\bar{A}'\oplus\bar{B}\xrightarrow{\reflectbox{c}^{\varphi}}\bar{B}\to\emptyset$. Define now a map $A'\to\bar{A}'\oplus B$ by $A'(\bar{A}'\oplus B)=A'\bar{A}'c(\bar{A}'\oplus B)+A'Bc(\bar{A}'\oplus B)$. Then commutativity holds in the diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
A'\colon & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d, Rightarrow] & A' \arrow[r] \arrow[d] & \bar{A}' \arrow[r] \arrow[d] & \emptyset \\
 & & \dot{B} \arrow[r] & \bar{A}'\oplus B \arrow[r] & \bar{A}'\oplus\bar{B}, &
\end{tikzcd}$$

and so by Lemma 3×2.4 we get $A'(\bar{A}'\oplus B)=\mathrm{kernel}(\bar{A}'\oplus B)(\bar{A}'\oplus\bar{B})\reflectbox{c}^{\varphi}\bar{B}$. On the other hand we have

$$\begin{aligned}
&\{A\bar{A}\bar{A}'c(\bar{A}'\oplus B)+ABc(\bar{A}'\oplus B)\}\cdot(\bar{A}'\oplus B)(\bar{A}'\oplus\bar{B})\reflectbox{c}^{\varphi}\bar{B}\\
&\qquad=-A\bar{A}\bar{A}'\bar{B}+AB\bar{B}=-A\bar{A}\bar{B}+\bar{A}B\bar{B}=0,
\end{aligned}$$

and so there is a unique map $A\to A'$ such that

$$AA'(\bar{A}'\oplus B)=A\bar{A}\bar{A}'c(\bar{A}'\oplus B)+ABc(\bar{A}'\oplus B),$$
$$\text{i.e., } AA'\bar{A}'=A\bar{A}\bar{A}' \text{ and } AA'B=AB.$$

Thus it remains only to show the commutativity $\dot{A}AA'=\dot{A}\dot{B}A'$. This follows from

$$\begin{aligned}
\dot{A}AA'(\bar{A}'\oplus B)&=\dot{A}A\bar{A}\bar{A}'c(\bar{A}'\oplus B)+\dot{A}ABc(\bar{A}'\oplus B)\\
&=0+\dot{A}\dot{B}Bc(\bar{A}'\oplus B)=\dot{A}\dot{B}(\bar{A}'\oplus B)=\dot{A}\dot{B}A'(\bar{A}'\oplus B),
\end{aligned}$$

for $A'(\bar{A}'\oplus B)$ is an injection.

LEMMA 3×2.7. *Assume* (Q2*). *In the* $(3\times2')$-*diagram suppose* $A'\bar{A}'=\mathrm{cokernel}\,\dot{B}A'$, $B\bar{B}=\mathrm{cokernel}\,\dot{B}B$. *If* $A'B$ *is a special injection, then* $\bar{A}'\bar{B}$ *is*

also a special injection.

PROOF. Let $B \to L$ be the cokernel of the special injection $A'B$. Then by Lemma 3×2.1, (iii)* we obtain the commutative diagram

$$\begin{tikzcd}
 & \dot{B} \arrow[r, Rightarrow] \arrow[d] & \dot{B} \arrow[d] & & \\
\emptyset \arrow[r] & A' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
 & \bar{A}' \arrow[r] & \bar{B} \arrow[r] & L &
\end{tikzcd},$$

where $\bar{B}L = \operatorname{cokernel} \bar{A}'\bar{B}$. Thus by Lemma 3×2.5* the sequence $\emptyset \to \bar{A}' \to \bar{B} \to L \to \emptyset$ is special exact.

LEMMA 3×2.8. *Assume* (Q2) *and* (Q2*). *In the* (3×2′)-*diagram suppose* $A'\bar{A}' = \operatorname{cokernel} \dot{B}A'$ *and* B *is special exact. Then* A' *is special exact.*

PROOF. Imbed B and $\bar{A}'\bar{B}$ in a commutative diagram

$$\begin{tikzcd}
A_1\colon & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d, Rightarrow] & A_1 \arrow[r] \arrow[d] & \bar{A}' \arrow[r] \arrow[d] & \emptyset \\
B\colon & \emptyset \arrow[r] & \dot{B} \arrow[r] & B \arrow[r] & \bar{B} \arrow[r] & \emptyset
\end{tikzcd}$$

with special exact rows. Then by Lemma 3×2.6 we obtain the commutative diagram

$$\begin{tikzcd}
 & & \dot{B} \arrow[r] \arrow[d, Rightarrow] & A' \arrow[r] \arrow[d] & \bar{A}' \arrow[d, Rightarrow] & \\
A_1\colon & \emptyset \arrow[r] & \dot{B} \arrow[r] & A_1 \arrow[r] & \bar{A}' \arrow[r] & \emptyset.
\end{tikzcd}$$

In virtue of (Q2*) we can now apply Lemma 3×2.4, (iii)* to conclude that A' is special exact.

COROLLARY. *If* $\psi \circ \varphi$ *is a special injection and if* φ *admits the cokernel, then* φ *is a special injection.*

2.2. In continuation to the preceding paragraph we prove some more lemmas on special exact sequences. The following commutative diagrams will be referred to as the (3×3)-diagram and the (3×3−1)-diagram respectively:

$$\begin{tikzcd}
 & & B\colon & E\colon & A\colon & \\
 & & \emptyset \arrow[d] & \emptyset \arrow[d] & \emptyset \arrow[d] & \\
\dot{X}\colon & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d] & \dot{E} \arrow[r] \arrow[d] & \dot{A} \arrow[r] \arrow[d] & \emptyset \\
X\colon & \emptyset \arrow[r] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \emptyset \\
\bar{X}\colon & \emptyset \arrow[r] & \bar{B} \arrow[r] \arrow[d] & \bar{E} \arrow[r] \arrow[d] & \bar{A} \arrow[r] \arrow[d] & \emptyset, \\
 & & \emptyset & \emptyset & \emptyset &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & E\colon & A\colon & \\
 & & & \emptyset \arrow[d] & \emptyset \arrow[d] & \\
\dot{X}\colon & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & \dot{E} \arrow[r] \arrow[d] & \dot{A} \arrow[r] \arrow[d] & \emptyset \\
X\colon & \emptyset \arrow[r] & B \arrow[r] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \emptyset \\
 & & & \bar{A} \arrow[r, Rightarrow] \arrow[d] & \bar{A} \arrow[d] & \\
 & & & \emptyset & \emptyset &
\end{tikzcd},$$

where the rows are special exact, and the columns may or may not be exact. The S-category $(\mathcal{A}, \mathcal{S})$ is now assumed to be regular.

LEMMA 3×3.3. *In the* $(3\times 3-1)$*-diagram we have:*

(i) *E is exact if and only if A is exact.*

(ii) *If A is direct exact, E is special exact.*

(iii) *If E is special exact, A is special exact.*

(iv) *The converse of* (iii) *holds if* $(\mathcal{A}, \mathcal{S})$ *is quasi-abelian.*

PROOF. Ad (i): Suppose E is exact. Then we have $A\bar{A}=\text{cokernel}\,\dot{A}A$ by Lemma 3×2.1, (iii)*, $\dot{A}A=\text{kernel}\,\bar{A}A$ by Lemma 3×2.5, and so A is exact. Likewise the converse follows from Lemma 3×2.1, (ii)* and Lemma 3×2.4, (ii)

Ad (ii): Firstly E is exact by (i). Let $A\to\dot{A}$, $\bar{A}\to A$ be maps giving $(\dot{A}A, \bar{A}A, A\dot{A}, A\bar{A})$: $A\approx\dot{A}\oplus\bar{A}$. Then the direct sum sequence of $\emptyset\to\emptyset\to\dot{A}\Rightarrow\dot{A}\to\emptyset$ and E can be written as

$$\tilde{E}:\ \emptyset\to\dot{E}\to\dot{A}\oplus E\to A\to\emptyset,$$

where $\dot{E}(\dot{A}\oplus E)=\dot{E}Ec(\dot{A}\oplus E)$ and $(\dot{A}\oplus E)A=(\dot{A}\oplus E)\text{ɔ}\dot{A}A+(\dot{A}\oplus E)\text{ɔ}E\bar{A}A$. We now define $E\to(\dot{A}\oplus E)$ by $E(\dot{A}\oplus E)=EA\dot{A}c(\dot{A}\oplus E)+Ec(\dot{A}\oplus E)$. Then we get $BE(\dot{A}\oplus E)$ $=BEA\dot{A}c(\dot{A}\oplus E)+BEc(\dot{A}\oplus E)=0+B\dot{E}E(\dot{A}\oplus E)=B\dot{E}(\dot{A}\oplus E)$ and $E(\dot{A}\oplus E)A=EA\dot{A}A+E\bar{A}A=EA\cdot(A\dot{A}A+A\bar{A}A)=EA$. Thus the diagram

$$\begin{array}{llcccccccc} X: & \emptyset & \to & B & \longrightarrow & E & \longrightarrow & A & \to & \emptyset \\ & & & \downarrow & & \downarrow & & \Downarrow & & \\ \tilde{E}: & \emptyset & \to & \dot{E} & \to & \dot{A}\oplus E & \to & A & \to & \emptyset \end{array}$$

is commutative, and so $\tilde{E}$ is special by Lemma 3×2.8*. Then Lemma 3×2.8 applied to the canonical injection $E\to\tilde{E}$ shows that B is also special.

Ad (iii): This is an immediate consequence of (i) and Lemma 3×2.8*.

Ad (iv): If A is special exact, then $E\bar{A}=EA\bar{A}$ is a special projection by (Q0). Being exact by (i), E is then special exact. Q. E. D.

Since the $(3\times 3-1)$-diagram can be reconstructed from any special projection $E\to A$ and any proper projection $A\to\bar{A}$, we get:

COROLLARY. $\mathrm{PJ}\mathcal{A}\circ\mathcal{S}\mathrm{J}=\mathrm{PJ}\mathcal{A}$, $\mathcal{S}\mathrm{I}\circ\mathrm{PI}\mathcal{A}=\mathrm{PI}\mathcal{A}$,
$\mathrm{DJ}\mathcal{A}\circ\mathcal{S}\mathrm{J}=\mathcal{S}\mathrm{J}$, $\mathcal{S}\mathrm{I}\circ\mathrm{DI}\mathcal{A}=\mathcal{S}\mathrm{I}$.

In particular if the S-category $(\mathcal{A}, \mathrm{P}\mathcal{A})$ is regular, then it is automatically quasi-abelian.

LEMMA 3×2.9. *In the commutative diagram*

$$(\beta\ulcorner X)\qquad \begin{array}{llcccccccc} X: & \emptyset & \to & B & \xrightarrow{\psi} & E & \xrightarrow{\varphi} & A & \to & \emptyset \\ \downarrow & & & \downarrow{\scriptstyle\beta} & & \downarrow & & \Downarrow & & \\ X': & \emptyset & \to & B' & \to & E' & \to & A & \to & \emptyset \end{array}$$

suppose X, X' are special exact. Define $B\to B'\oplus E$, $B'\oplus E\to E'$ by $B(B'\oplus E)$ $=-BB'c(B'\oplus E)+BEc(B'\oplus E)$, $(B'\oplus E)E'=(B'\oplus E)\text{ɔ}B'E'+(B'\oplus E)\text{ɔ}EE'$. Then commutativity holds in the diagram

$$
\begin{tikzcd}[row sep=small]
\beta\wedge\psi: & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & B'\oplus E \arrow[r] \arrow[d, "\text{ɔ}"] & E' \arrow[r] \arrow[d] & \emptyset \\
X: & \emptyset \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & \emptyset
\end{tikzcd}
$$

and the sequence $(\beta\wedge\psi)$ *is special exact.*

Before giving the proof we shall give a few remarks. By Lemmas 3×2.6*, 3×2.4, (iii) the special exeact sequence X' is (upto equivalence) uniquely determined by X and β. This is why we have designated the first diagram as $(\beta\ulcorner X)$.

The notation $(\beta\wedge\psi)$ comes from the same reason. We shall often write $\beta\bigcirc X$ for the sequence X'. Similarly $X\bigcirc\alpha$, $(X\lrcorner\alpha)$, $\varphi\vee\beta$ are defined for $\alpha: A'\to A$.

PROOF OF LEMMA 3×2.9. Take the direct sum sequence $\tilde{X}: \emptyset\to B'\oplus B\to B'\oplus E \to A\to\emptyset$ of $\emptyset\to B'\Rightarrow B'\to\emptyset\to\emptyset$ and X. Recall that the sequence $\emptyset\to B\xrightarrow{c_\beta} B'\oplus B\xrightarrow{\text{ɔ}_\beta} B'\to\emptyset$, given by $Bc_\beta(B'\oplus B)=-BB'c(B'\oplus B)+Bc(B'\oplus B)$ and $(B'\oplus B)\text{ɔ}_\beta B'=(B'\oplus B)\text{ɔ}B' +(B'\oplus B)\text{ɔ}BB'$, is direct exact. Since commutativity holds in the diagram

$$
\begin{tikzcd}[row sep=small]
 & & & \beta\wedge\psi: & & \\
 & & \emptyset \arrow[d] & \emptyset \arrow[d] & & \\
 & & B \arrow[r, Rightarrow] \arrow[d, "c_\beta"] & B \arrow[d] & & \\
X: & \emptyset \arrow[r] & B'\oplus B \arrow[r] \arrow[d, "\text{ɔ}_\beta"] & B'\oplus E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
X': & \emptyset \arrow[r] & B' \arrow[r] & E' \arrow[r] & A \arrow[r] & \emptyset
\end{tikzcd}\quad,
$$

the sequence $\beta\wedge\psi$ is special exact by Lemma 3×3.3, (ii)*. Commutativity in the second diagram is obvious.

LEMMA 3×2.10. *In the commutative diagram*

$$
\begin{tikzcd}[row sep=small]
\dot{X}: & \emptyset \arrow[r] & \dot{B} \arrow[r, "\dot{\psi}"] \arrow[d, "\dot{\beta}"] & \dot{E} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
X: & \emptyset \arrow[r] & B \arrow[r, "\psi"] & E \arrow[r] & A \arrow[r] & \emptyset
\end{tikzcd}
$$

with special exact rows suppose $\dot{B}B$ *is a proper injection with the cokernel* $B\to\bar{B}$. *Then there is a unique map* $E\to\bar{B}$ *such that* $BE\bar{B}=B\bar{B}$, $\dot{E}E\bar{B}=0$. *Moreover the sequence* $\emptyset\to\dot{E}\to E\to\bar{B}\to\emptyset$ *is exact. If* $\dot{B}B$ *is a special injection, then* $\dot{E}E$ *is also a special injection.*

PROOF. Firstly commutativity holds in the diagram

$$
\begin{tikzcd}[row sep=small]
 & & & \dot{E} \arrow[r, Rightarrow] \arrow[d, "c"] & \dot{E} \arrow[d] & \\
\dot{\beta}\wedge\psi: & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d, Rightarrow] & B\oplus\dot{E} \arrow[r] \arrow[d, "\text{ɔ}"] & \dot{E} \arrow[r] & \emptyset \\
 & \emptyset \arrow[r] & \dot{B} \arrow[r, "-\dot{\beta}"] & B \arrow[r] & \bar{B} \arrow[r] & \emptyset .
\end{tikzcd}
$$

Hence there is a unique map $E\to\bar{B}$ with which the diagram remains commutative. This commutativity $(B\oplus\dot{E})E\bar{B}=(B\oplus\dot{E})\text{ɔ}B\bar{B}$ means that on the two components B, $\dot{E}$ we have $BE\bar{B}=B\bar{B}$ and $\dot{E}E\bar{B}=0$. So the first part is proved. Next Lemma

3×2, (i)* applied to the so completed diagram shows $E\bar{B}=\text{cokernel}\,\dot{E}E$. To prove $\dot{E}E=\text{kernel}\,E\bar{B}$, let $\bar{C}\to E$ be any map with $\bar{C}\bar{B}=0$, and use (Q2) to obtain the commutative diagram

$$\begin{tikzcd}
& \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d, Rightarrow] & C \arrow[r] \arrow[d] & \bar{C} \arrow[r] \arrow[d] & \emptyset \\
\dot{\beta}\wedge\dot{\psi}: & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d, Rightarrow] & B\oplus\dot{E} \arrow[r] \arrow[d, "c"] & E \arrow[r] \arrow[d] & \emptyset \\
& \emptyset \arrow[r] & \dot{B} \arrow[r, "-\dot{\beta}"] & B \arrow[r] & \bar{B} \arrow[r] & \emptyset
\end{tikzcd}$$

where the first two rows are special exact. Since $\bar{C}E\bar{B}=0$, we take $0:\bar{C}\to B$, and apply Lemma 3×2.2 to obtain a map $\bar{C}\to C$ such that $\bar{C}C\bar{C}=\bar{C}e\bar{C}$, $\bar{C}C(B\oplus\dot{E})\text{ɔ}B=0$. Define $\bar{C}\to\dot{E}$ by $\bar{C}\dot{E}=\bar{C}C(B\oplus\dot{E})\text{ɔ}\dot{E}$. Then we get $\bar{C}E=\bar{C}C\bar{C}E=\bar{C}C(B\oplus\dot{E})E$ $=\bar{C}C(B\oplus\dot{E})\text{ɔ}BE+\bar{C}C(B\oplus\dot{E})\text{ɔ}\dot{E}E=0+\bar{C}\dot{E}E$. On the other hand $\dot{E}E$ is an injection by Lemma 3×2.1, (iv). This together with $E\bar{B}=\text{cokernel}\,\dot{E}E$ shows that the sequence $\emptyset\to\dot{E}\to E\to\bar{B}\to\emptyset$ is exact. Finally in view of the commutative diagram

$$\begin{tikzcd}
\emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & \bar{B} \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
\emptyset \arrow[r] & \dot{E} \arrow[r] & E \arrow[r] & \bar{B} \arrow[r] & \emptyset
\end{tikzcd}$$

the last part is obvious from Lemma 3×2.8*.

LEMMA 3×2.11. *Assume* $(\mathcal{A},\mathcal{S})$ *is quasi-abelian. In the commutative diagram*

$$\begin{tikzcd}
\dot{X}: & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d] & \dot{E} \arrow[r] \arrow[d] & \dot{A} \arrow[r] \arrow[d, "\alpha"] & \emptyset \\
X: & \emptyset \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & \emptyset
\end{tikzcd}$$

with special exact rows suppose $\dot{A}A$, $\dot{B}B$ *are special injections. Then* $\dot{E}E$ *is also a special injection.*

PROOF. Consider the commutative diagram $(X\lrcorner\alpha)$;

$$\begin{tikzcd}
X\bigcirc\alpha: & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & E' \arrow[r] \arrow[d] & \dot{A} \arrow[r] \arrow[d] & \emptyset \\
X: & \emptyset \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & \emptyset .
\end{tikzcd}$$

By Lemma 3×3.3, (iv) $E'E$ is a special injection. On the other hand by Lemma 3×2.6 there is a unique map $\dot{E}\to E'$ such that the diagram

$$\begin{tikzcd}
\dot{X}: & \emptyset \arrow[r] & \dot{B} \arrow[r] \arrow[d] & \dot{E} \arrow[r] \arrow[d] & \dot{A} \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
X\bigcirc\alpha: & \emptyset \arrow[r] & B \arrow[r] & E' \arrow[r] & \dot{A} \arrow[r] & \emptyset
\end{tikzcd}$$

is commutative and $\dot{E}E'E=\dot{E}E$. By Lemma 3×2.10, $\dot{E}E'$ is a special injection, and so $\dot{E}E$ is a special injection by (Q0*).

LEMMA 3×3.4. *Assume* $(\mathcal{A},\mathcal{S})$ *is quasi-abelian. In the* (3×3)*-diagram suppose* $\boldsymbol{B}$ *is special exact.*

(i) *If* $\boldsymbol{E}$ *is special exact,* $\boldsymbol{A}$ *is specsal exact.*

(ii) *If* $\boldsymbol{A}$ *is special exact and* $\dot{\boldsymbol{E}}\boldsymbol{E}\bar{\boldsymbol{E}}=0$, *then* $\boldsymbol{E}$ *is special exact.*

PROOF. Ad (i): Consider the commutative diagram $(E\lrcorner\bar{B}\bar{E})$:

$$\begin{array}{ccccccccc} E'\colon & \emptyset & \to & \dot{E} & \to & E' & \to & \bar{B} & \to \emptyset \\ & & & \Downarrow & & \downarrow & & \downarrow & \\ E\colon & \emptyset & \to & \dot{E} & \to & E & \to & \bar{E} & \to \emptyset . \end{array}$$

By Lemma 3×3.3, (iv) the sequence $\emptyset\to E'\to E\to\bar{A}\to\emptyset$ $(E\bar{A}=E\bar{E}\bar{A})$ is special exact. By Lemma 3×2.6 there is a unique map $B\to E'$ such that the diagram

$$\begin{array}{ccccccccc} B\colon & \emptyset & \to & \dot{B} & \to & B & \to & \bar{B} & \to \emptyset \\ & & & \downarrow & & \downarrow & & \Downarrow & \\ E'\colon & \emptyset & \to & \dot{E} & \to & E' & \to & \bar{B} & \to \emptyset \end{array}$$

is commutative and $BE'E=BE$. Further by Lemma 3×2.10 there is a unique map $E'\to\dot{A}$ such that $\dot{E}E'\dot{A}=\dot{E}\dot{A}$, $BE'\dot{A}=0$. Moreover the sequence $\emptyset\to B\to E'\to \dot{A}\to\emptyset$ is special exact. We now assert that commutativity holds in the diagram

$$\begin{array}{ccccccccc} & & & & \emptyset & & & & \\ & & & & \downarrow & & & & \\ \emptyset & \to & B & \to & E' & \to & \dot{A} & \to & \emptyset \\ & & \Downarrow & & \downarrow & & \downarrow & & \\ \emptyset & \to & B & \to & E & \to & A & \to & \emptyset , \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bar{A} & \Rightarrow & \bar{A} & & \\ & & & & \downarrow & & & & \\ & & & & \emptyset & & & & \end{array}$$

from which will follow exactness of A by Lemma 3×3, (iii). We have already $BE'E=BE$ and $E\bar{A}=E\bar{E}\bar{A}=EA\bar{A}$. So it remains only to prove $E'EA=E'\dot{A}A$. Now that the two rows in the diagram are exact, there is a map $\alpha'\colon \dot{A}\to A$ with $E'\dot{A}\alpha'A=E'EA$. Then we get $\dot{E}\dot{A}\alpha'A=\dot{E}E'\dot{A}\alpha'A=\dot{E}E'EA=\dot{E}EA$. Since the commutative diagram $E\to A$ determines the map $\dot{A}\to A$, we get $\dot{A}\alpha'A=\dot{A}A$, and so the above diagram is commutative. Hence A is special exact.

Ad (ii): By Lemma 3×2.11, $\dot{E}E$ is a special injection. Let $E\to\bar{E}'$ be the cokernel of $\dot{E}E$. Then by the part (i) we get the (3×3)-diagram with $\bar{E}$ replaced by $\bar{E}'$, where rows and columns are all special exact. Since $\dot{E}E\bar{E}=0$, there is a map $\bar{E}'\to\bar{E}$ such that $E\bar{E}'\bar{E}=E\bar{E}$. From $B\bar{B}\bar{E}'\bar{E}=BE\bar{E}'\bar{E}=BE\bar{E}=B\bar{B}\bar{E}$ follows $\bar{B}\bar{E}'\bar{E}=\bar{B}\bar{E}$, and from $E\bar{E}'\bar{E}\bar{A}=E\bar{E}\bar{A}=EA\bar{A}=E\bar{E}'\bar{A}$ follows $\bar{E}'\bar{E}\bar{A}=\bar{E}'\bar{A}$. Thus commutativity holds in the diagram

$$\begin{array}{ccccccccc} \emptyset & \to & \bar{B} & \to & \bar{E}' & \to & \bar{A} & \to & \emptyset \\ & & \Downarrow & & \downarrow & & \Downarrow & & \\ \emptyset & \to & \bar{B} & \to & \bar{E} & \to & \bar{A} & \to & \emptyset , \end{array}$$

and so $\bar{E}'\bar{E}$ is an equivalence map by Lemma 3×2.4, (iv). Hence E is special exact.

2.3. In this paragraph we shall give some remarks on how to check whether a given S-category $(\mathcal{A}, \mathcal{S})$ is regular (or quasi-abelian). Out of the lemmas hitherto established we get necessary conditions for the S-category to be regular (or

quasi-abelian). Some combinations of them will constitute criteria. For example the following two conditions make clearly a criterion for the S-category to be quasi-abelian:

(Q) Any special exact sequence $\emptyset\to B\to E\to A\to\emptyset$ and any special injection $\dot{A}\to A$ can be imbedded in a commutative diagram

$$\begin{array}{ccccccccc} \emptyset & \to & B & \to & \dot{E} & \to & \dot{A} & \to & \emptyset \\ & & \Downarrow & & \downarrow & & \downarrow & & \\ \emptyset & \to & B & \to & E & \to & A & \to & \emptyset \end{array}$$

where rows are special exact and $\dot{E}E$ is special (it is *a fortiori* an injection).

(Q*): The dual of (Q).

If $(\mathcal{A}, \mathcal{S})$ is regular then by the corollary to Lemma 3×3.3 we have the composition rules:

(Q0.1) $\mathrm{DJ}\mathcal{A}\circ\mathcal{S}\mathrm{J}=\mathcal{S}\mathrm{J}$.

(Q0.1*) $\mathcal{S}\mathrm{I}\circ\mathrm{DI}\mathcal{A}=\mathcal{S}\mathrm{I}$.

Suppose $(\mathcal{A}, \mathcal{S})$ satisfies (Q0.1). Given any special exact sequence $X: \emptyset\to B\to E\to A\to\emptyset$ and any direct injection $\dot{A}\to A$ with the cokernel $A\to\bar{A}$, we take the kernel $\dot{E}\to E$ of the special projection $EA\bar{A}$. This gives rise to the commutative diagram

$$\begin{array}{ccccccccc} & & & & \emptyset & & \emptyset & & \\ & & & & \downarrow & & \downarrow & & \\ & & B & \to & \dot{E} & \to & \dot{A} & & \\ & & \Downarrow & & \downarrow & & \downarrow & & \\ X: \ \emptyset & \to & B & \to & E & \to & A & \to & \emptyset \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bar{A} & \Rightarrow & \bar{A} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \emptyset & & \emptyset & & \end{array}$$

where $B\dot{E}=\text{kernel}\,\dot{E}\dot{A}$ by Lemma 3×2.1, (iii), and $\dot{E}\dot{A}$ is a projection by Lemma 3×2.3. Hence (Q0.1) and the following weakened form of Lemma 3×3.3, (iii)* imply (Q2), namely:

(Q3) In the commutative diagram

$$\begin{array}{ccccccccc} \emptyset & \to & \dot{E} & \to & E & \to & \bar{A} & \to & \emptyset \\ & & \downarrow & & \downarrow & & \Downarrow & & \\ \emptyset & \to & \dot{A} & \to & A & \to & \bar{A} & \to & \emptyset \end{array}$$

suppose the first row is special exact, second is direct exact, and EA is a special projection. Then $\dot{E}\dot{A}$ is special.

Thus the combination (Q0.1), (Q0.1*), (Q3), (Q3*) gives a regularity criterion, and the combination (Q0), (Q0*), (Q3), (Q3*) gives a quasi-abelian criterion.

There are other ways to guarantee special exactness of $\emptyset\to B\to\dot{E}\to\dot{A}\to\emptyset$. Consider the following weakened form of (Q2*), namely:

(Q2.1*) Any special exact sequence $X: \emptyset\to B\to E\to A\to\emptyset$ and any map $B\to B'$ can

be extended to a map $X \to X'$ in $\mathcal{E}^1\mathcal{A}$.
This condition implies exactness of $\emptyset \to B \to \dot{E} \to \dot{A} \to \emptyset$. In fact let $\dot{E} \to \dot{C}$ be any map with $B\dot{E}\dot{C}=0$. Imbed the special exact sequence $\emptyset \to \dot{E} \to E \to \bar{A} \to \emptyset$ and $\dot{E}\dot{C}$ in a commutative diagram

$$\begin{array}{rccccccccc} E: & \emptyset & \to & \dot{E} & \to & E & \to & \bar{A} & \to & \emptyset \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ C: & \emptyset & \to & \dot{C} & \to & C & \to & \bar{C} & \to & \emptyset \end{array}$$

with exact rows. Because of $BEC=B\dot{E}EC=B\dot{E}\dot{C}C=0$ there is a map $A \to C$ such that $EAC=EC$. Then from $\dot{E}\dot{A}AC\bar{C}=\dot{E}EAC\bar{C}=\dot{E}EC\bar{C}=\dot{E}\dot{C}C\bar{C}=0$ follows $\dot{A}AC\bar{C}=0$, for $\dot{E}\dot{A}$ is a projection. Consequently there is a map $\dot{A} \to \dot{C}$ such that $\dot{A}\dot{C}C=\dot{A}AC$. Finally from $\dot{E}\dot{A}\dot{C}C=\dot{E}\dot{A}AC=\dot{E}EAC=\dot{E}EC=\dot{E}\dot{C}C$ follows $\dot{E}\dot{A}\dot{C}=\dot{E}\dot{C}$, and so we get $\dot{E}\dot{A}=\operatorname{cokernel} B\dot{E}$. We now add to this the following weakened form of the corollary to Lemma 3×2.8, namely:

(Q4) If ψ, $\psi \circ \varphi$ are special injections and if φ is a proper injection, then φ is a special injection.

This condition guarantees that the exact sequence E is special. So we get the implication (Q0.1)$\cap$(Q2.1)$\cap$(Q4)$\to$(Q2). Note that (Q4) is trivial for the S-category $(\mathcal{A}, \mathrm{P}\mathcal{A})$. Note also that (Q2.1*) holds if there is a regular S-category $(\mathcal{A}, \mathcal{S}_1)$ such that $\mathcal{S}_1 \supset \mathcal{S}$, in particular if $\mathcal{A}$ is a quasi-abelian category. (Q2.1*) holds also when $\mathcal{A}$ has enough injectives.

PROPOSITION 2.1. *If* $(\mathcal{A}, \mathcal{S}_j)$ *are regular* (*quasi-abelian*), *then the* S-*category* $(\mathcal{A}, \cap \mathcal{S}_j)$ *is also regular* (*quasi-aberian*).
In fact the conditions (Q0.1), (Q0.1*), (Q0), (Q0*), (Q3), (Q3*) are hereditary to intersections.

Let $(\mathcal{A}, \mathcal{S})$, $(\mathcal{A}', \mathcal{S}')$ be S-categories, and let $f: \mathcal{A} \to \mathcal{A}'$ be an additive functor. We denote by $f^{\#}\mathcal{S}'$ the class of maps in $\mathcal{A}$ consisting of proper maps $\varphi: A \to B$ such that the exact sequences $\emptyset \to \operatorname{Ker}\varphi \to A \to \operatorname{Im}\varphi \to \emptyset$, $\emptyset \to \operatorname{Im}\varphi \to B \to \operatorname{Coker}\varphi \to \emptyset$ are carried over by f to $\mathcal{S}'$-special exact sequences. Obviously $(\mathcal{A}, f^{\#}\mathcal{S}')$, $(\mathcal{A}, \mathcal{S} \cap f^{\#}\mathcal{S}')$ are S-categories. We shall say that f is *half* $\mathcal{S}$-*exact* if for any $\mathcal{S}$-special exact sequence $\emptyset \to B \to E \to A \to \emptyset$ the sequence $f(B) \to f(E) \to f(A)$ (arrows reversed if f is contravariant) is exact. The following proposition provides us with various examples of quasi-abelian S-categories:

PROPOSITION 2.2. *If* $(\mathcal{A}, \mathcal{S})$, $(\mathcal{A}', \mathcal{S}')$ *are regular* (*quasi-abelian*) *and if* f *is half* $\mathcal{S}$-*exact, then the* S-*category* $(\mathcal{A}, \mathcal{S} \cap f^{\#}\mathcal{S}')$ *is regular* (*quasi-abelian*).

PROOF. We shall only prove the quasi-abelian case, since the regular case is proved similarly. Without loss of generality we may assume f is covariant. Let $X: \emptyset \to B \to E \to A \to \emptyset$ be an $\mathcal{S} \cap f^{\#}\mathcal{S}'$-special exact sequence, and let $\dot{A} \to A$ be an

$\mathcal{S} \cap f^{\#}\mathcal{S}'$-special injection. Since $(\mathcal{A}, \mathcal{S})$ is quasi-abelian these can be imbedded in a commutative diagram

$$\begin{tikzcd}
 & & E: & A: & \\
 & & \emptyset \arrow[d] & \emptyset \arrow[d] & \\
\dot{X}: \ \emptyset \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & \dot{E} \arrow[r] \arrow[d] & \dot{A} \arrow[r] \arrow[d] & \emptyset \\
X: \ \emptyset \arrow[r] & B \arrow[r] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \emptyset \\
 & & \bar{A} \arrow[r, Rightarrow] \arrow[d] & \bar{A} \arrow[d] & \\
 & & \emptyset & \emptyset &
\end{tikzcd},$$

where $\dot{X}$, E are $\mathcal{S}$-special exact. Since $f(EA)$, $f(A\bar{A})$ are $\mathcal{S}'$-special projections, so is $f(E\bar{A})$. On the other hand $f(B\dot{E}E)=f(BE)$ is an $\mathcal{S}'$-special injection and by hypothesis $f(B)\to f(\dot{E})\to f(\dot{A})$ is exact. Therefore the proper map $f(B\dot{E})$ is an $\mathcal{S}'$-special injection by the corollary to Lemma 3×2.8, and it is the kernel of $f(\dot{E}\dot{A})$. Then by Lemma 3×2.1, (iv) $f(\dot{E}E)$ is an injection. Now the sequence $f(\dot{E})\to f(E)\to f(\bar{A})$ being exact and $f(E\bar{A})$ being an $\mathcal{S}'$-special projection, the sequence $f(E)$ is $\mathcal{S}'$-special exact. Consequently $f(\dot{X})$ is $\mathcal{S}'$-special exact by Lemma 3×3.3, (iii)*. This shows that E, $\dot{X}$ are $\mathcal{S} \cap f^{\#}\mathcal{S}'$-special, and so $(\mathcal{A}, \mathcal{S} \cap f^{\#}\mathcal{S}')$ satisfies (Q). Dually (Q*) is verified, and the proof is completed.

§ 3. Similarity classification and composition product.

3.0. Let $(\mathcal{A}, \mathcal{S})$ be an S-category. We shall denote be $\mathcal{S}\mathcal{E}^n\mathcal{A}$ $(n \geqq 1)$ the full subcategory of the additive category $\mathcal{E}^n\mathcal{A}$ consisting of special exact sequences. Namely an object X in $\mathcal{S}\mathcal{E}^n\mathcal{A}$ means a special exact sequence in $\mathcal{A}$ of the form

$$X: \ \emptyset \to B \to E_n \to E_{n-1} \to \cdots \to E_1 \to A \to \emptyset,$$

a map $\xi: X \to X'$ in $\mathcal{S}\mathcal{E}^n\mathcal{A}$ means a commutative diagram

$$\begin{tikzcd}
X: \ \emptyset \arrow[r] & B \arrow[r] \arrow[d, "\beta"] & E_n \arrow[r] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & \emptyset \\
X': \ \emptyset \arrow[r] & B' \arrow[r] & E'_n \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r] & A' \arrow[r] & \emptyset
\end{tikzcd}$$

with special exact rows, and the composition (addition) of maps in $\mathcal{S}\mathcal{E}^n\mathcal{A}$ is given by composing (adding) the vertical constituents term by term. Direct sums in $\mathcal{S}\mathcal{E}^n\mathcal{A}$ are given by the direct sum of special exact sequences, and thus $\mathcal{S}\mathcal{E}^n\mathcal{A}$ is an additive category. We define additive covariant functors $\mathcal{S}E^n_-$, $\mathcal{S}E^n_+$: $\mathcal{S}\mathcal{E}^n\mathcal{A} \to \mathcal{A}$ by $\mathcal{S}E^n_-(X)=A$, $\mathcal{S}E^n_-(\xi)=\alpha$, $\mathcal{S}E^n_+(X)=B$, $\mathcal{S}E^n_+(\xi)=\beta$. Also $\mathcal{S}E^n$ will denote the obvious covariant functor $\mathcal{S}\mathcal{E}^n\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ having the first component $\mathcal{S}E^n_-$ and the second component $\mathcal{S}E^n_+$. For a pair of objects (A, B) in $\mathcal{A}$, $\mathrm{EXT}^n{}_{(\mathcal{A}, \mathcal{S})}(A, B)$ will stand for the totality of objects X in $\mathcal{S}\mathcal{E}^n\mathcal{A}$ such that $\mathcal{S}E^n(X)=(A, B)$. For two objects X_1, X_2 in $\mathrm{EXT}^n{}_{(\mathcal{A}, \mathcal{S})}(A, B)$ we shall write $X_1 \rightrightarrows X_2$ if there exists a map $\xi: X_1 \to X_2$ in $\mathcal{S}\mathcal{E}^n\mathcal{A}$ such that $\mathcal{S}\mathcal{E}^n(\xi)=(e_A, e_B)$. The binary relation $\rightrightarrows$ is reflexive

and transitive. It generates an equivalence relation $\sim$ in $\mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(A,B)$, which we shall call *similarity*. Namely $X\sim X'$ if there is a finite series of relations

$$X \rightleftarrows X_1 \rightrightarrows X_2 \rightleftarrows \cdots \rightrightarrows X_{2h} \rightleftarrows X' .$$

The objects X in $\mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ now fall apart into a collection $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ of similarity classes $[X]$. For the case $(\mathcal{A}, \mathcal{S})=({}_{\Lambda}\mathcal{M}, {}_{\Lambda}\mathcal{M})$ we have established in [10] a certain 1-1-correspondence between $\mathrm{Ext}^n{}_{\Lambda}(A, B)$ and $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ using projectives or injectives. We shall now give functorial and additive structure directly to $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ without recourse to projectives or injectives, but under the assumption that $(\mathcal{A}, \mathcal{S})$ is regular.

In what follows, putting an eye on the situation $\mathcal{S}E^n$: $\mathcal{S}\mathcal{E}^n\mathcal{A}\to\mathcal{A}\times\mathcal{A}$, we consider a more general similarity classification. Namely we consider the situation defined as follows:

DEFINITION. *A span over a pair of categories $(\mathcal{A}, \mathcal{B})$ consists in a covariant functor S: $\mathcal{X}\to\mathcal{A}\times\mathcal{B}$ from a category $\mathcal{X}$.*

S_-: $\mathcal{X}\to\mathcal{A}$, S_+: $\mathcal{X}\to\mathcal{B}$ will stand for the components of S. For a pair (A, B) of objects A in $\mathcal{A}$ and B in $\mathcal{B}$ we write $S^{-1}(A, B)$ for the totality of objects X in $\mathcal{X}$ with $S(X)=(A, B)$. For two objects X_1, X_2 in $S^{-1}(A, B)$ we write $X_1 \rightrightarrows X_2$ if there exists a map ξ: $X_1\to X_2$ in $\mathcal{X}$ such that $S(\xi)=(e_A, e_B)$. *Similarity* in $\mathcal{X}$ is the equivalence relation $\sim$ generated by $\rightrightarrows$, by which $S^{-1}(A,B)$ is divided into a collection $\bar{S}(A, B)$ of similarity classes.

3.1. Let S: $\mathcal{X}\to\mathcal{A}\times\mathcal{B}$ be a span. Let X be an object in $S^{-1}(A, B)$, and α: $A'\to A$ a map in $\mathcal{A}$. By the *cotranslation of X by α* we mean an object $X^\alpha\in S^{-1}(A', B)$ together with a map ξ^α: $X^\alpha\to X$ in $\mathcal{X}$ with the following properties: (i) $S(\xi^\alpha)=(\alpha, e_B)$; (ii) For any maps α': $A''\to A'$ in $\mathcal{A}$ and ξ'': $X''\to X$ in $\mathcal{X}$ with $S_-(X'')=A''$, $S_-(\xi'')=\alpha\circ\alpha'$ there is a unique map ξ': $X''\to X^\alpha$ in $\mathcal{X}$ such that $\xi^\alpha\circ\xi'=\xi''$, $S_-(\xi')=\alpha'$.

Also by the *translation of X by β*: $B\to B'$ we mean an object ${}_\beta X\in S^{-1}(A, B')$ together with a map ${}_\beta\xi$: $X\to{}_\beta X$ in $\mathcal{X}$ with the properties: (i*) $S({}_\beta\xi)=(e_A, \beta)$; (ii*) For any maps β': $B'\to B''$ in $\mathcal{B}$ and ξ'': $X\to X''$ in $\mathcal{X}$ with $S_+(X'')=B''$, $S_+(\xi'')=\beta'\circ\beta$ there is a unique map ξ': ${}_\beta X\to X''$ in $\mathcal{X}$ such that $\xi'\circ{}_\beta\xi=\xi''$, $S_+(\xi')=\beta'$. (N. B. The author [10] once used the term 'translation' in a different sense. In fact the present concept is motivated by consideration of universal completions of partly given 'translations' in a 'translation category'.)

These two notions are dual to each other. Thereby the dual of a span S: $\mathcal{X}\to\mathcal{A}\times\mathcal{B}$ should be understood as the obvious span S^*: $\mathcal{X}^*\to\mathcal{B}^*\times\mathcal{A}^*$ and the duality principle on spans should be interpreted accordingly. The following properties

of translations and contranslations are more or less obvious; We shall only supply the proofs to some of them:

(3.1.1.) *Let $X^\alpha \to X$, $X_1{}^\alpha \to X$ be the cotranslations of $X \in S^{-1}(A, B)$ by $\alpha: A' \to A$. Then there is a unique map $X_1{}^\alpha \to X^\alpha$ in $\mathfrak{X}$ such that $X_1{}^\alpha X^\alpha X = X_1{}^\alpha X$, $S(X_1{}^\alpha X^\alpha) = (A'eA', BeB)$. Moreover $X_1{}^\alpha X^\alpha$ is an equivalence map.*

In fact the first part is immediate from the definition. For the second part we have also a unique map $X^\alpha \to X_1{}^\alpha$ in $\mathfrak{X}$ such that $X^\alpha X_1{}^\alpha X = X^\alpha X$, $S(X^\alpha X_1{}^\alpha) = (A'eA', BeB)$. Then the composition $X^\alpha X_1{}^\alpha X^\alpha$ gives $X^\alpha X_1{}^\alpha X^\alpha X = X^\alpha X$, $S(X^\alpha X_1{}^\alpha X^\alpha) = (A'eA', BeB)$. Since $X^\alpha e X^\alpha$ also gives $X^\alpha e X^\alpha X = X^\alpha X$, $S(X^\alpha e X^\alpha) = (A'eA', BeB)$, and since such a map $X^\alpha \to X^\alpha$ must be unique, we get $X^\alpha X_1{}^\alpha X^\alpha = X^\alpha e X^\alpha$. Similarly we obtain $X_1{}^\alpha X^\alpha X_1{}^\alpha = X_1{}^\alpha e X_1{}^\alpha$.

(3.1.2.) *If $X^\alpha \to X$ is an equivalence map in $\mathfrak{X}$ with $S(X^\alpha X) = (\alpha, e_B)$ (α is necessarily an equivalence map in $\mathcal{A}$), then it is the cotranslation of X by α. In particular XeX is the cotranslation of X by AeA. The cotarnslation of X by AeA is necessarily an equivalence map in $\mathfrak{X}$.*

(3.1.3.) *If $X^\alpha \to X$ is the cotranslation of $X \in S^{-1}(A, B)$ by $\alpha: A' \to A$, and if $(X^\alpha)^{\alpha'} \to X^\alpha$ is the cotranslation of X^α by $\alpha': A'' \to A'$, then $(X^\alpha)^{\alpha'} X^\alpha X$ is the cotranslation of X by $\alpha \circ \alpha'$.*

In fact let $A''' \to A''$ be any map in $\mathcal{A}$, and let $X''' \to X$ be any map in $\mathfrak{X}$ with $S_-(X''') = A'''$, $S_-(X'''X) = A'''A''A'A$. Then there is a unique map $X''' \to X^\alpha$ such that $X'''X^\alpha X = X'''X$, $S_-(X'''X^\alpha) = A'''A''A'$, and further there is a unique map $X''' \to (X^\alpha)^{\alpha'}$ such that $X'''(X^\alpha)^{\alpha'} X^\alpha = X'''X^\alpha$, $S_-(X'''(X^\alpha)^{\alpha'}) = A'''A''$. So we have $X'''(X^\alpha)^{\alpha'} X^\alpha X = X'''X^\alpha X = X'''X$. Uniqueness of $X''' \to (X^\alpha)^{\alpha'}$ under the conditions $X'''(X^\alpha)^{\alpha'} X^\alpha X = X'''X$, $S_-(X'''(X^\alpha)^{\alpha'}) = A'''A''$ is obvious.

(3.1.4.) *Let $X \to {}_\beta X$ be the translation of $X \in S^{-1}(A, B)$ by $\beta: B \to B'$, and let $X'^\alpha \to X'$ be the cotranslation of $X' \in S^{-1}(A', B')$ by $\alpha: A \to A'$. If there exists a map $\xi: X \to X'$ in $\mathfrak{X}$ with $S(\xi) = (\alpha, \beta)$, then we have ${}_\beta X \simeq X'^\alpha$.*

In fact there is a unique map $X \to X'^\alpha$ such that $XX'^\alpha X' = XX'$, $S(XX'^\alpha) = (AeA, BB')$, and then there is a unique map ${}_\beta X \to X'^\alpha$ such that $X_\beta X X'^\alpha = XX'^\alpha$, $S({}_\beta X X'^\alpha) = (AeA, B'eB')$.

(3.1.5.) *If cotranslations always exist in the span, then the cotranslation $X^\alpha \to X$ of $X \in S^{-1}(A, B)$ by an equivalence map $\alpha: A' \to A$ in $\mathcal{A}$ is necessarily an equivalence map in $\mathfrak{X}$.*

In fact let $X_1 \to X^\alpha$ be the contranslation of X^α by $\alpha^{-1}: A \to A'$, and let $X_1{}^\alpha \to X_1$ be the contranslation of X_1 by α. Then by (3.1.3), $X_1 X^\alpha X$ is the contranslation of X by e_A and $X_1{}^\alpha X_1 X^\alpha$ is the cotranslation of X^α by $e_{A'}$. Therefore $X_1 X^\alpha X$, $X_1{}^\alpha X_1 X^\alpha$ are equivalence maps in $\mathfrak{X}$. Let $X \to X_1$, $X^\alpha \to X_1{}^\alpha$ be the respective inverse

maps, and define $X\to X^{\alpha}$ by $XX^{\alpha}=XX_1X^{\alpha}$. Then we obtain $XX^{\alpha}X=XX_1X^{\alpha}X$ $=XX_1\cdot X_1X^{\alpha}X=XeX$ and $X^{\alpha}XX^{\alpha}=X^{\alpha}X_1^{\alpha}\cdot X_1^{\alpha}X_1X^{\alpha}\cdot X^{\alpha}XX^{\alpha}=X^{\alpha}X_1^{\alpha}X_1\cdot X_1X^{\alpha}X\cdot$ $XX_1\cdot X_1X^{\alpha}=X^{\alpha}X_1^{\alpha}X_1eX_1X^{\alpha}=X^{\alpha}X_1^{\alpha}\cdot X_1^{\alpha}X_1X^{\alpha}=X^{\alpha}eX^{\alpha}$. Thus $X^{\alpha}X$ is an equivalence map with the inverse map XX^{α}.

In the cotranslation $X^{\alpha}\to X$ of $X\in S^{-1}(A, B)$ by $\alpha: A'\to A$ the object X^{α} is unique upto equivalence over $(e_{A'}, e_B)$. If $(\mathcal{A}, \mathcal{S})$ is a regular S-category, then in the span $\mathcal{S}E^1$: $\mathcal{S}\mathcal{E}^1\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ the map $(X\lrcorner\alpha): X\bigcirc\alpha\to X$ is the cotranslation of X by α, because of Lemma 3×2.6. Dually $(\beta\ulcorner X): X\to\beta\bigcirc X$ is the translation of X by β. So in general we shall write $X\bigcirc\alpha$ for X^{α}, and $\beta\bigcirc X$ for ${}_{\beta}X$ in the translation $X\to{}_{\beta}X$ of X by β. Further we shall say that a span $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$ is *regular* if translations $X\to\beta\bigcirc X$ and cotranslations $X\bigcirc\alpha\to X$ always exist. Clearly we have:

PROPOSITION 3.1. *For an* S-*category* $(\mathcal{A}, \mathcal{S})$ *the span* $\mathcal{S}E^1$: $\mathcal{S}\mathcal{E}^1\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ *is regular if and only if* $(\mathcal{A}, \mathcal{S})$ *is regular.*

The same symbol $\bigcirc$ was employed also for the composed sequence $Y\bigcirc X$: $\cdots\to F\to E\to\cdots$ of two exact sequences $X: \emptyset\to B\to E\to\cdots$ and $Y: \cdots\to F\to B\to\emptyset$. The reason is that as will be shown later these operations under the symbol $\bigcirc$ can be treated in a unified manner. Here we note the followings: Firstly $\bigcirc$ gives a pairing

$$\bigcirc:\ \mathrm{EXT}^q{}_{(\mathcal{A},\mathcal{S})}(B, C)\times\mathrm{EXT}^p{}_{(\mathcal{A},\mathcal{S})}(A, B)\to\mathrm{EXT}^{p+q}{}_{(\mathcal{A},\mathcal{S})}(A, C),$$

which is associative, namely for $X\in\mathrm{EXT}^p{}_{(\mathcal{A},\mathcal{S})}(A, B)$, $Y\in\mathrm{EXT}^q{}_{(\mathcal{A},\mathcal{S})}(B, C)$, and $W\in\mathrm{EXT}^r{}_{(\mathcal{A},\mathcal{S})}(C, D)$ we have $W\bigcirc(Y\bigcirc X)=(W\bigcirc Y\bigcirc)X$. On the other hand for $X\in\mathrm{EXT}^1{}_{(\mathcal{A},\mathcal{S})}(A, B)$, $Y\in\mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(B, C)$, and $\alpha\in\mathrm{Hom}(A', A)$ the cotranslation $X\bigcirc\alpha\to X$ joined with YeY gives a map $Y\bigcirc(X\bigcirc\alpha)\to Y\bigcirc X$, which is clearly the cotranslation of $Y\bigcirc X$ by α. In other words upto equivalence over $(e_{A'}, e_C)$ we have $Y\bigcirc(X\bigcirc\alpha)=(Y\bigcirc X)\bigcirc\alpha$. Dually for $X\in\mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$, $Y\in\mathrm{EXT}^1{}_{(\mathcal{A},\mathcal{S})}(B, C)$ and $\gamma\in\mathrm{Hom}(C, C')$ we have $(\gamma\bigcirc Y)\bigcirc X=\gamma\bigcirc(Y\bigcirc X)$ upto equivalence over $(e_A, e_{C'})$. In particular we have:

PROPOSITION 3.2. *For a regular* S-*category* $(\mathcal{A}, \mathcal{S})$ *the spans* $\mathcal{S}E^n$: $\mathcal{S}\mathcal{E}^n\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ $(n=2, 3, \cdots)$ *are also regular.*

3.2. Let $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$ be a span such that contranslations $X\bigcirc\alpha\to X$ always exist.

PROPOSITION 3.3. *For each map* $\alpha: A'\to A$ *in* $\mathcal{A}$ *and for each object* B *in* $\mathcal{B}$ *there is a unique mapping*

$$\circ\alpha:\ \bar{S}(A, B)\to\bar{S}(A', B)\qquad (\textit{we write } [X]\circ\alpha \textit{ for } \circ\alpha([X]))$$

such that for $X\in S^{-1}(A, B)$, $X'\in S^{-1}(A', B)$ *we have* $[X]\circ\alpha=[X']$ *if there exists a map* $\xi: X'\to X$ *with* $S(\xi)=(\alpha, e_B)$. *Moreover we have*

(3.2.1) $[X]\circ e_A=[X]$,

(3.2.2) $[X]\circ(\alpha\circ\alpha')=([X]\circ\alpha)\circ\alpha'$.

PROOF. For $X\in S^{-1}(A, B)$ and $\alpha: A'\to A$ we take the cotranslation $X\bigcirc\alpha\to X$ of X by α. Since the cotranslation is unique upto equivalence over $(e_{A'}, e_B)$, $[X\bigcirc\alpha]$ does not depend on the choice of the cotranslation. If there exists a map $X'\to X$ over (α, e_B), then we have a map $X'\to X\bigcirc\alpha$ over $(e_{A'}, e_B)$, and so $[X']=[X\bigcirc\alpha]$. Now if a map $X\to X_1$ gives $X\simeq X_1$, then the composition of $X\bigcirc\alpha\to X$ and $X\to X_1$ is over (α, e_B), and so $[X\bigcirc\alpha]=[X_1\bigcirc\alpha]$. This shows that $[X\bigcirc\alpha]\in\bar{S}(A', B)$ depends only on $[X]$ and α. Thus it is obligatory and legitimate to put $[X]\circ\alpha=[X\bigcirc\alpha]$. (3.2.1) and (3.2.2) follow from (3.1.2) and (3.1.3). This completes the proof.

The dual of Proposition 3.3 will be for a span $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$ such that translations $X\to\beta\bigcirc X$ always exist, asserting that each map $\beta: B\to B'$ induces a unique mapping

$$\beta\circ:\ \bar{S}(A, B)\to\bar{S}(A, B')$$

such that existence of a map $X\to X'$ over (e_A, β) implies $\beta\circ[X]=[X']$. We have only to set $\beta\circ[X]=[\beta\bigcirc X]$, and we get the formulas:

(3.2.1*) $e_B\circ[X]=[X]$.

(3.2.2*) $(\beta'\circ\beta)\circ[X]=\beta'\circ(\beta\circ[X])$.

Let now $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$ be a regular span, so that we have the mappings $\circ\alpha$, $\beta\circ$ defined by $[X]\circ\alpha=[X\bigcirc\alpha]$, $\beta\circ[X]=[\beta\bigcirc X]$.

PROPOSITION 3.4. *For $X\in S^{-1}(A, B)$, $X'\in S^{-1}(A', B')$ and for maps $\alpha: A\to A'$, $\beta: B\to B'$ we have $\beta\circ[X]=[X']\circ\alpha$ if there exists a map $\xi: X\to X'$ in $\mathcal{X}$ such that $S(\xi)=(\alpha, \beta)$. Moreover for $[X]\in\bar{S}(A, B)$, $\alpha: A\to A'$, $\beta: B\to B'$ we have*

(3.2.3) $\beta\circ([X]\circ\alpha)=(\beta\circ[X])\circ\alpha$.

PROOF. The first part is an immediate consequence of (3.1.4). As for the second part let $X\bigcirc\alpha\to X$ be the cotranslation of X by α, and let $X\to\beta\bigcirc X$ be the translation of X by β. Then the composed map $X\bigcirc\alpha\to\beta\bigcirc X$ is over (α, β), and so by the first part we get $\beta\circ[X\bigcirc\alpha]=[\beta\bigcirc X]\circ\alpha$. This verifies the formula (3.2.3).

Combining Propositions 3.3, 3.3*, 3.4 we get:

CLASSIFICATION THEOREM. *Let $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$ be a regular span. Then the system of collections $\bar{S}(A, B)$ is extended to a functor $\bar{S}(a, b)$ of two variables a contravariant in $\mathcal{A}$, b covariant in $\mathcal{B}$, and the values are collections and element-wise mappings. Thereby the functorial structure is characterized by the property that for $X\in S^{-1}(A, B)$, $X'\in S^{-1}(A', B')$ and for maps $\alpha: A\to A'$, $\beta: B\to B'$ we have $\bar{S}(A, \beta)[X]=\bar{S}(\alpha, B')[X']$ if there exists a map $\xi: X\to X'$ in $\mathcal{X}$ such taat $S(\xi)=(\alpha, \beta)$.*

3.3. By an *additive span* we mean a span $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$ where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$ are

additive categories and the covariant functors S_-, S_+ are additive. For an S-category $(\mathcal{A}, \mathcal{S})$ the spans $\mathcal{S}E^n$: $\mathcal{S}\mathcal{E}^n\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ are additive. Moreover in $\mathcal{S}E^n$ the direct sum of $X_\nu\in \mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(A_\nu, B_\nu)$ $(\nu=1,2)$ can be so chosen that it covers any preassigned direct sums $A_1\oplus A_2$, $B_1\oplus B_2$. Formally this may not hold in an arbitrary additive span. One could have modified the definition of additive spans in postulating the above liftibility of direct sums, but it is not difficult to see that the liftibility of direct sums holds in any regular additive span. Take any direct sum X of X_1 and X_2. Then we have canonical equivalence maps $\alpha: A_1\oplus A_2\to S_-(X)$ $\beta: S_+(X)\to B_1\oplus B_2$, and it will suffice to replace X by $\beta\bigcirc(X\bigcirc\alpha)$ noticing (3.1.5). In speaking of direct sums in $\mathcal{X}$ in a regular additive span $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$, we shall mean those which cover the preassigned direct sums in $\mathcal{A}$ and in $\mathcal{B}$. Namely for $X_\nu\in S^{-1}(A_\nu, B_\nu)$ $(\nu=1,2)$ and the direct sums $A_1\oplus A_2$, $B_1\oplus B_2$, the direct sum $X_1\oplus X_2$ is required to be such that

$$\begin{aligned} &X_1\oplus X_2\in S^{-1}(A_1\oplus A_2, B_1\oplus B_2),\\ &S(X_\nu c(X_1\oplus X_2))=(A_\nu c(A_1\oplus A_2), B_\nu c(B_1\oplus B_2)),\\ &S((X_1\oplus X_2)\ni X_\nu)=((A_1\oplus A_2)\ni A_\nu, (B_1\oplus B_2)\ni B_\nu) \qquad (\nu=1,2). \end{aligned}$$

We aim at the following:

ADDITIVE CLASSIFICATION THEOREM. *Let $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$ be a regular additive span. Then every collection $\bar{S}(A, B)$ is not empty, and addition in each $\bar{S}(A, B)$ can be uniquely defined so that the functor $\bar{S}(a, b)$ is additive either in a (i.e. $\bar{S}(\alpha_1+\alpha_2, B)[X]=\bar{S}(\alpha_1, B)[X]+\bar{S}(\alpha_2, B)[X]$) or in b. It is automatically additive in both variables, and $\bar{S}(\alpha, B)$, $\bar{S}(A, \beta)$ respect the addition. The addition is realized by*

(3.3.1) $[X_1]+[X_2]=\nabla_B\circ[X_1\oplus X_2]\circ\Delta_A$,

where $X_1, X_2\in S^{-1}(A, B)$, $X_1\oplus X_2\in S^{-1}(A\oplus A, B\oplus B)$, Δ_A is the diagonal map $Ac_1(A\oplus A)+Ac_2(A\oplus A)$, and ∇_B is the codiagonal map $(B\oplus B)\ni_1 B+(B\oplus B)\ni_2 B$. $\bar{S}(A, B)$ becomes an additive group with

(3.3.2) $[(\emptyset\bigcirc 0^A)\oplus(0_B\bigcirc\emptyset)]=0\in\bar{S}(A, B)$,

(3.3.3) $-[X]=(-e_B)\circ[X]=[X]\circ(-e_A)$,

where $\emptyset\in S^{-1}(\emptyset,\emptyset)$ is the neutral object in $\mathcal{X}$, 0^A is the trivial map $A\emptyset$ in $\mathcal{A}$, and 0_B is the trivial map $\emptyset B$ in $\mathcal{B}$. $\bar{S}(a, b)$ becomes thus an $\mathcal{A}^*$-$\mathcal{B}$-module.

COROLLARY. *Let $(\mathcal{A}, \mathcal{S})$ be a regular* S-*category. For $n\geqq 1$ the system of collections $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ of similarity classes $[X]$ of special exact sequences of the form*

$$X:\ \emptyset\to B\to E_n\to E_{n-1}\to\cdots\to E_1\to A\to\emptyset$$

is made into an $\mathcal{A}^*$-$\mathcal{A}$-module, i.e. an additive functor of two variables in $\mathcal{A}$

contravariant in the first entry, covariant in the second entry, and with values in the category $\mathfrak{M}$ *of additive groups and homomorphisms. The functorial structure is characterized by that the existence of a commutative diagram*

$$\begin{array}{rl} X: & \emptyset \to B \to E_n \to \cdots \to E_1 \to A \to \emptyset \\ & \qquad \downarrow\beta \quad\ \downarrow \qquad\qquad \downarrow \quad\ \downarrow\alpha \\ X': & \emptyset \to B' \to E'_n \to \cdots \to E'_1 \to A' \to \emptyset \end{array}$$

implies $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A,\beta)[X]=\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(\alpha,B')[X']$. *The additive structure is characterized by that the functor is additive in both variables. The zero element in* $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A,B)$ *is represented by the trivial exact sequence*

$$O^n(A,B): \qquad \emptyset \to B \xrightarrow{c} A\oplus B \xrightarrow{\supset} A \to \emptyset \qquad (n=1),$$

$$\emptyset \to B \Rightarrow B \xrightarrow{0} A \Rightarrow A \to \emptyset \qquad (n=2),$$

$$\emptyset \to B \Rightarrow B \to \emptyset \to \cdots \to \emptyset \to A \Rightarrow A \to \emptyset \qquad (n\geqq 3).$$

The negative $-[X]$ *of* $[X]$ *is represented by changing the sign of any odd number of maps among* $B\to E_n$, $E_n\to E_{n-1}$, $\cdots$, $E_1\to A$. *Addition in* $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A,B)$ *is realized by* (3.3.1).

Note that for $\mathcal{E}^1{}_A\mathfrak{M}$ the addition (3.3.1) reduces to the 'Baer multiplication' in [4, p. 290].

3.4. PROOF OF ADDITIVE CLASSIFICATION THEOREM. Firstly $(\emptyset\bigcirc 0^A)\oplus(0_B\bigcirc\emptyset)$ should be an object in $S^{-1}(A,B)$, and so $\bar{S}(A,B)$ is not empty. To prove uniqueness of addition it suffices to show that (3.3.1) is obligatory if $\bar{S}(a,b)$ is additive in a. Denote the cononical maps $Ac_\nu(A\oplus A)$, $(A\oplus A)\supset_\nu A$, $Bc_\nu(B\oplus B)$, $(B\oplus B)\supset_\nu B$ by α_ν, $\bar{\alpha}_\nu$, β_ν, $\bar{\beta}_\nu$ $(\nu=1,2)$ respectively. Then we must have

$$\begin{aligned}\nabla_B\circ[X_1\oplus X_2]\circ\Delta_A&=\nabla_B\circ[X_1\oplus X_2]\circ(\alpha_1+\alpha_2)\\ &=\nabla_B\circ[X_1\oplus X_2]\circ\alpha_1+\nabla_B\circ[X_1\oplus X_2]\circ\alpha_2\,.\end{aligned}$$

Now the canonical injection $X_\nu\to X_1\oplus X_2$ is over (α_ν,β_ν) $(\nu=1,2)$, and so we get $\beta_\nu\circ[X_\nu]=[X_1\oplus X_2]\circ\alpha_\nu$. Hence by (3.2.1*), (3.2.2*) we get

$$\nabla_B\circ[X_1\oplus X_2]\circ\Delta_A=\nabla_B\circ\beta_1\circ[X_2]+\nabla_B\circ\beta_2\circ[X_1]=e_B\circ[X_1]+e_B\circ[X_2]=[X_1]+[X_2].$$

Thus uniqueness of addition is verified.

Next we assert that addition is well defined by (3.3.1). First we fix the direct sums $A\oplus A$, $B\oplus B$. Then $X_1^1\simeq X_1^2$ implies $X_1^1\oplus X_2\simeq X_1^2\oplus X_2$, and so $[X_1\oplus X_2]$ does not depend on the choice of the representative X_1 from $[X_1]$. In the same way $[X_1\oplus X_2]$ does not depend on X_2, but on $[X_2]$. This shows also that $[X_1\oplus X_2]$ does not depend on the choice of the direct sum $X_1\oplus X_2$ (as long as it covers the direct sums $A\oplus A$, $B\oplus B$). Next we fix X_1, X_2 and take any direct sums $(\alpha'_1,\alpha'_2,\bar{\alpha}'_1,\bar{\alpha}'_2): A^2\approx A\oplus A$, $(\beta'_1,\beta'_2,\bar{\beta}'_1,\bar{\beta}'_2): B^2\approx B\oplus B$. Let $(\xi_1,\xi_2,\bar{\xi}_1,\bar{\xi}_2): X^2\approx X_1\oplus X_2$ be the direct sum over these direct sums. The canonical equivalence map $\xi: X^2\to X_1\oplus X_2$

is obviously over the canonical equivalence maps $\alpha: A^2 \to A\oplus A$, $\beta: B^2 \approx B\oplus B$. So we get $\beta\circ[X]=[X_1\oplus X_2]\circ\alpha$, and therefore $[X^2]=\beta^{-1}\circ[X_1\oplus X_2]\circ\alpha$. Because of $\alpha=\alpha_1\circ\bar{\alpha}_1'+\alpha_2\circ\bar{\alpha}_2'$, $\beta^{-1}=\beta_1'\circ\bar{\beta}_1+\beta_2'\circ\bar{\beta}_2$ we obtain $\alpha\circ(\alpha_1'+\alpha_2')=\alpha_1+\alpha_2=\Delta_A$, $(\bar{\beta}_1'+\bar{\beta}_2')\circ\beta^{-1}=\bar{\beta}_1+\bar{\beta}_2=\nabla_B$. Consequently we have

$$(\bar{\beta}_1'+\bar{\beta}_2')\circ[X^2]\circ(\alpha_1'+\alpha_2')=\nabla_B\circ[X_1\oplus X_2]\circ\Delta_A,$$

which shows that $\nabla_B\circ[X_1\oplus X_2]\circ\Delta_A$ does not depend on the choice of the direct sums $A\oplus A$, $B\oplus B$. Hence (3.3.1) is legitimate.

We now give some preparatory formulas. Let $\xi_\nu: X_\nu\bigcirc\alpha_\nu \to X_\nu$ $(\nu=1, 2)$ be the cotranslations of $X_1, X_2\in S^{-1}(A, B)$ by $\alpha_1, \alpha_2\in\mathrm{Hom}(A', A)$. Then the direct sum $\xi_1\oplus\xi_2: (X_1\bigcirc\alpha_1)\oplus(X_2\bigcirc\alpha_2)\to X_1\oplus X_2$ is over $(\alpha_1\oplus\alpha_2, e_{B\oplus B})$. (In fact it is the cotranslation of $X_1\oplus X_2$ by $\alpha_1\oplus\alpha_2$.) This proves

(3.4.1) $$[(X_1\bigcirc\alpha_1)\oplus(X_2\bigcirc\alpha_2)]=[X_1\oplus X_2]\circ(\alpha_1\oplus\alpha_2).$$

For $X\in S^{-1}(A, B)$ the codiagonal map $\nabla_X: X\oplus X\to X$ is over (∇_A, ∇_B). So we have

(3.4.2) $$[X]\circ\nabla_A=\nabla_B\circ[X\oplus X].$$

The next two formulas are obvious:

(3.4.3) $$\alpha_1+\alpha_2=\nabla_A\circ(\alpha_1\oplus\alpha_2)\circ\Delta_{A'} \qquad (\alpha_1, \alpha_2\in\mathrm{Hom}(A',A));$$

(3.4.4) $$\Delta_A\circ\alpha=(\alpha\oplus\alpha)\circ\Delta_{A'} \qquad (\alpha\in\mathrm{Hom}(A', A)).$$

Using those formulas the distributivity formulas are checked easily:

(3.4.5) $$[X]\circ(\alpha_1+\alpha_2)=[X]\circ\alpha_1+[X]\circ\alpha_2;$$

(3.4.5*) $$(\beta_1+\beta_2)\circ[X]=\beta_1\circ[X]+\beta_2\circ[X].$$

In fact we have

$$\begin{aligned}[X]\circ(\alpha_1+\alpha_2)&=[X]\circ\nabla_A(\alpha_1\oplus\alpha_2)\circ\Delta_{A'} && (3.4.3)\\ &=\nabla_B\circ[X\oplus X]\circ(\alpha_1\oplus\alpha_2)\circ\Delta_{A'} && (3.4.2)\\ &=\nabla_B\circ[(X\bigcirc\alpha_1)\oplus(X\bigcirc\alpha_2)]\circ\Delta_{A'} && (3.4.1)\\ &=[X\bigcirc\alpha_1]+[X\bigcirc\alpha_2] && (3.1.3)\\ &=[X]\circ\alpha_1+[X]\circ\alpha_2.\end{aligned}$$

(3.4.6) $$([X_1]+[X_2])\circ\alpha=[X_1]\circ\alpha_1+[X]\circ\alpha_2;$$

(3.4.6*) $$\beta\circ([X_1]+[X_2])=\beta\circ[X_1]+\beta\circ[X_2].$$

In fact we have

$$\begin{aligned}([X_1]+[X_2])\circ\alpha&=\nabla_B\circ[X_1\oplus X_2]\circ\Delta_A\circ\alpha && (3.3.1)\\ &=\nabla_B\circ[X_1\oplus X_2]\circ(\alpha\oplus\alpha)\circ\Delta_{A'} && (3.4.4)\\ &=\nabla_B\circ[(X_1\bigcirc\alpha)\oplus(X_2\bigcirc\alpha)]\circ\Delta_{A'} && (3.4.1)\\ &=[X_1\bigcirc\alpha]+[X_2\bigcirc\alpha] && (3.3.1)\\ &=[X_1]\circ\alpha+[X_2]\circ\alpha.\end{aligned}$$

Now the commutativity and associativity of the addition in $\bar{S}(A, B)$ may be checked directly starting from (3.3.1), but these can also be derived from the distributivity formulas. Consider for example the associativity. With the canonical

equivalence maps counted for, there will be no ambiguity in writting

$$X_1\oplus X_2\oplus X_3\in S^{-1}(A\oplus A\oplus A,\ B\oplus B\oplus B)\quad (X_1,\ X_2,\ X_2\in S^{-1}(A,\ B)).$$

Let $\alpha_1, \alpha_2, \alpha_3$ stand for the canonical injections $A\to A\oplus A\oplus A$, $\beta_1, \beta_2, \beta_3$ for the canonical injections $B\to B\oplus B\oplus B$, and $\bar{\beta}$ for the sum of the three canonical projections. Since $\bar{\beta}\circ\beta_\nu=e_B$ we have $[X_\nu]=\bar{\beta}\circ\beta_\nu\circ[X_\nu]$ ($\nu=1, 2, 3$). On the other hand in virtue of the canonical injection $X_\nu\to X_1\oplus X_2\oplus X_3$ over (α_ν, β_ν) we have $\beta_\nu\circ[X_\nu]=[X_1\oplus X_2\oplus X_3]\circ\alpha_\nu$, and so we get

$$[X_\nu]=\bar{\beta}\circ[X_1\oplus X_2\oplus X_3]\circ\alpha_\nu \qquad (\nu=1, 2, 3).$$

Thus in view of (3.4.5), addition of $[X_1]$, $[X_2]$, $[X_3]$ in $\bar{S}(A, B)$ corresponds to addition of $\alpha_1, \alpha_2, \alpha_3$ in $\mathrm{Hom}(A, A\oplus A\oplus A)$, and so the associativity

(3.4.7) $$([X_1]+[X_2])+[X_3]=[X_1]+([X_2]+[X_3])$$

is obvious. Likewise the commutativity

(3.4.8) $$[X_1]+[X_2]=[X_2]+[X_1]$$

is verified.

Finally put $\emptyset^A=\emptyset\bigcirc 0^A$, $\emptyset_B=0_B\bigcirc\emptyset$, $\emptyset_B^A=(\emptyset\bigcirc 0^A)\oplus(0_B\bigcirc\emptyset)$. $0_A^A=0_A\circ 0^A$ $(=0\in\mathrm{Hom}(A,A))$. Since the canonical injection $\emptyset^A\to\emptyset_B^A$ is over $(e_A, 0_B)$ we have $0_B\circ[\emptyset^A]=[\emptyset_B^A]$. Furthermore for any $X\in S^{-1}(A, B)$ we have $[X]\circ 0_A^A=[\emptyset_B^A]$, for the trivial map $\emptyset^A\emptyset$ in $\mathcal{X}$, being over $(0_A^A, 0_B)$, gives $0_B\circ[\emptyset^A]=[X]\circ 0_A^A$. Therefore using (3.4.5), we obtain

$$[X]+[\emptyset_B^A]=[X]\circ(e_A+0_A^A)=[X],$$
$$[X]+[X]\circ(-e_A)=[X]\circ 0_A^A=[\emptyset_B^A],$$

whereas the map $-e_X: X\to X$ over $(-e_A, -e_B)$ gives $(-e_B)\circ[X]=[X]\circ(-e_A)$. This completes the proof of the theorem.

3.5. Let $(\mathcal{A}, \mathcal{S})$ be a regular S-category, and let

$$X:\ \emptyset\to B\to E_p\to\cdots\to E_1\to A\to\emptyset,$$
$$Y:\ \emptyset\to C\to F_q\to\cdots\to F_1\to B\to\emptyset$$

be special exact sequences in $\mathcal{A}$ representing $[X]\in\mathrm{Ext}^p{}_{(\mathcal{A},\mathcal{S})}(A,B)$ and $[Y]\in\mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(B, C)$. Then the composed sequence

$$Y\bigcirc X:\ \emptyset\to C\to F_q\to\cdots\to F_1\to E_p\to\cdots\to E_1\to A\to\emptyset \qquad (F_1E_p=F_1BE_p),$$

being special exact, represents an element $[Y\bigcirc X]\in\mathrm{Ext}^{p+q}{}_{(\mathcal{A},\mathcal{S})}(A, C)$. It is readily seen that $[Y\bigcirc X]$ depends only on $[X]$ and $[Y]$, so that $\bigcirc$ induces a pairing

(3.5.1) $$\circ:\ \mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(B, C)\times\mathrm{Ext}^p{}_{(\mathcal{A},\mathcal{S})}(A, B)\to\mathrm{Ext}^{p+q}{}_{(\mathcal{A},\mathcal{S})}(A, C),$$
$$[Y]\circ[X]=[Y\bigcirc X],$$

which we call the *composition* or the *composition product*. We now interpret this product in terms of regular spans, in generalizing the product to a pairing

$$\circ:\ \bar{T}(B,C)\times\bar{S}(A,B)\to\overline{T*S}(A,C), \tag{3.5.2}$$

where S is a span over $(\mathcal{A},\mathcal{B})$, T a span over $(\mathcal{B},\mathcal{C})$, and $\overline{T*S}$ is a span over $(\mathcal{A},\mathcal{C})$ constructed in a certain way from S and T. It turns out that the operations so far defined under the symbol $\circ$ are unified in this generalized composition product (3.5.2). Then naturality, balancedness, and associativity of (3.5.1) are merely special cases of associativity of (3.5.2). Also bilinearity of (3.5.1) will be derived from bilinearity of (3.5.2) where S, T are additive regular spans. Note that the product defined in [10] was written in the reversed order. We have changed it to the present order to comply with the usual notation $\psi\circ\varphi$ of composition.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be categories, and let $S:\ \mathcal{X}\to\mathcal{A}\times\mathcal{B}$, $T:\ \mathcal{Y}\to\mathcal{B}\times\mathcal{C}$ be spans. Denote by $\mathcal{Y}*\mathcal{X}$ the subcategory of $\mathcal{Y}\times\mathcal{X}$ consisting of pairs (y,x) (of objects or of maps) such that $S_+(x)=T_-(y)$. We define a covariant functor $T*S:\ \mathcal{Y}*\mathcal{X}\to\mathcal{A}\times\mathcal{C}$, i.e. a span over $(\mathcal{A},\mathcal{C})$, by setting $T*S(y,x)=(S_-(x),\ T_+(y))$. This span $T*S$ will be called the *composed span* of S and T. If $X^\alpha\to X$ is the cotranslation of $X\in S^{-1}(A,B)$ by $\alpha:\ A'\to A$ with respect to S, then for any $Y\in T^{-1}(B,C)$ the map $(YeY,\ X^\alpha X)$ gives clearly the cotranslation of (Y,X) by α with respect to $T*S$. In other words up to equivalence we have $(Y,X\bigcirc\alpha)=(Y,X)\bigcirc\alpha$. Thus if cotranslations always exist in the span S, then the same holds also in $T*S$. In view of the duality $(T*S)^*=S^**T^*$, the dual result is that if translations always exist in T, then the same holds in $T*S$. In particular the composed span $T*S$ is regular if both S and T are regular.

Let X be an object in $S^{-1}(A,B)$, and $Y\in T^{-1}(B,C)$. Then the pair (Y,X) is an object in $(T*S)^{-1}(A,C)$. Obviously $X_1\simeq X_2$ in S implies $(Y,X_1)\simeq(Y,X_2)$ in $T*S$, and $Y_1\simeq Y_2$ in T implies $(Y_1,X)\simeq(Y_2,X)$. Therefore the similarity class $[Y,X]\in\overline{T*S}(A,C)$ of (Y,X) depends only on $[X]\in\bar{S}(A,B)$ and on $[Y]\in\bar{T}(B,C)$. So we obtain a pairing

$$\begin{gathered}\circ:\ \bar{T}(B,C)\times\bar{S}(A,B)\to\overline{T*S}(A,C),\\ [Y]\circ[X]=[Y,X].\end{gathered} \tag{3.5.2}$$

This will be called the (generalized) *composition product*. For an S-category $(\mathcal{A},\mathcal{S})$ the composed span $\mathcal{S}E^q*\mathcal{S}E^p:\mathcal{S}\mathcal{E}^q\mathcal{A}*\mathcal{S}\mathcal{E}^p\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ is not quite identical with $\mathcal{S}E^{p+q}:\mathcal{S}\mathcal{E}^{p+q}\mathcal{A}\to\mathcal{A}\times\mathcal{A}$. There the only difference is that the middle term B which is considered as specified in (Y,X) is hidden in $Y\bigcirc X$, and it can be restored only upto equivalence. Then it is clear that the obvious covariant functor $\mathcal{S}\mathcal{E}^q\mathcal{A}*\mathcal{S}\mathcal{E}^p\mathcal{A}\to\mathcal{S}\mathcal{E}^{p+q}\mathcal{A}$ sending (Y,X) to $Y\bigcirc X$ induces a 1-1-correspondence

$\overline{SE^q * SE^p}(A, C) \to \overline{SE^{p+q}}(A, C)$, through which we shall identify $\overline{SE^q * SE^p}(A, C)$ with $\overline{SE^{p+q}}(A, C)$. If $(\mathcal{A}, \mathcal{S})$ is regular, then this identification preserves the functorial and additive structure, and the functor $\overline{SE^q * SE^p}(a, c)$ is thus idendified with $\mathrm{Ext}^{p+q}{}_{(\mathcal{A}, \mathcal{S})}(a, c)$. In this sense (3.5.1) is a special case of (3.5.2).

Let $S: \mathcal{X} \to \mathcal{A} \times \mathcal{B}$, $T: \mathcal{Y} \to \mathcal{B} \times \mathcal{C}$, $U: \mathcal{Z} \to \mathcal{C} \times \mathcal{D}$ be spans. In the obvious manner we may consider $(U*T)*S$ as identical with $U*(T*S)$, so that we may write $U*T*S: \mathcal{Z}*\mathcal{Y}*\mathcal{X} \to \mathcal{A} \times \mathcal{D}$, where $\mathcal{Z}*\mathcal{Y}*\mathcal{X}$ is the subcategory of $\mathcal{Z} \times \mathcal{Y} \times \mathcal{X}$ consisting of triples (z, y, x) such that $S_+(x) = T_-(y)$, $T_+(y) = U_-(z)$. For $X \in S^{-1}(A, B)$, $Y \in T^{-1}(B, C)$, and $Z \in U^{-1}(C, D)$ both $[Z] \circ ([Y] \circ [X])$ and $([Z] \circ [Y]) \circ [X] \in \overline{U*T*S}(A, D)$ are represented by the same object (Z, Y, X) in $\mathcal{Z}*\mathcal{Y}*\mathcal{X}$. So we have the general associativity

(3.5.3) $$[Z] \circ ([Y] \circ [X]) = ([Z] \circ [Y]) \circ [X].$$

We now consider the span $E^0: \mathcal{E}^0\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ of maps in $\mathcal{A}$. An object X in $\mathcal{E}^0\mathcal{A}$ over a pair (A, B) in $\mathcal{A} \times \mathcal{A}$ means a map $\varphi: A \to B$ in $\mathcal{A}$. Let $\alpha: A' \to A$, $\beta: B \to B'$ be maps in $\mathcal{A}$. Then the maps in $\mathcal{E}^0\mathcal{A}$ given by the commutative diagrams

$$\begin{array}{rccc} \varphi\circ\alpha: & A' & \to & B \\ & \downarrow{\scriptstyle\alpha} & & \Downarrow \\ \varphi: & A & \to & B \end{array} \quad , \qquad \begin{array}{rccc} \varphi: & A & \to & B \\ & \Downarrow & & \downarrow{\scriptstyle\beta} \\ \beta\circ\varphi: & A & \to & B' \end{array}$$

are clearly the cotranslation of φ by α and the translation of φ by β respectively. Thus the span E^0 is regular. On the other hand in this span we have $\varphi_1 \simeq \varphi_2$ if and only if $\varphi_1 = \varphi_2$, and so the similarity class $[\varphi]$ of φ consists of φ alone. Since $[\varphi] \circ \alpha = [\varphi \bigcirc \alpha] = [\varphi \circ \alpha]$ and $\beta \circ [\varphi] = [\beta \bigcirc \varphi] = [\beta \circ \varphi]$ the functor $\bar{E}^0$ associated to this regular span E^0 may be identified with the Hom functor of $\mathcal{A}$. Thus for E^0: $\mathcal{E}^0\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ and $S: \mathcal{X} \to \mathcal{A} \times \mathcal{B}$ the composition product (3.5.2) can be written as

(3.5.4) $$\circ: \ \bar{S}(A, B) \times \mathrm{Hom}(A', A) \to \overline{S*E^0}(A', B).$$

Suppose cotranslations always exist in S. An object in $(S*E^0)^{-1}(A', B)$ means a pair of an object $X \in S^{-1}(A, B)$ and $\alpha: A' \to A$ for some A in $\mathcal{A}$. A map $\xi': (X_1, \alpha_1) \to (X_2, \alpha_2)$ in $\mathcal{X}*\mathcal{E}^0\mathcal{A}$ means a map $\xi: X_1 \to X_2$ in $\mathcal{X}$ paired with a commutative diagram

$$\begin{array}{rccl} \alpha_1: & A_1' & \to & A_1 = S_-(X_1) \\ & \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\alpha = S_-(\xi)} \\ \alpha_2: & A_2' & \to & A_2 = S_-(X_2), \end{array}$$

where we have $S*E^0(\xi') = (\alpha', \beta)$ $(\beta = S_+(\xi))$. If $\beta = e_B$, then we have $[X_2] \circ \alpha = [X_1]$, and so $[X_2] \circ \alpha_2 \circ \alpha' = [X_1] \circ \alpha_1$. This shows firstly that $[X] \circ \alpha \in \bar{S}(A', B)$ depends only on the similarity class $[X, \alpha] \in \overline{S*E^0}(A', B)$ of (X, α), and so we get a mapping

$$\theta: \ \overline{S*E^0}(A', B) \to \bar{S}(A', B),$$

$$\theta[X, \alpha]=[X]\circ\alpha=[X\bigcirc\alpha].$$

The above relation shows secondly that $\theta([X, \alpha]\circ\alpha')=(\theta[X, \alpha])\circ\alpha'$. Next to $X' \in S^{-1}(A', B)$ we associate the pair $(X', e_{A'})\in(S\ast E^0)^{-1}(A', B)$. Clearly the similarity class $[X', e_{A'}]\in\overline{S\ast E^0}(A', B)$ depends only on $[X']\in\bar{S}(A', B)$, and so we get a mapping

$$\theta': \ \bar{S}(A', B)\to\overline{S\ast E^0}(A', B), \qquad \theta'[X']=[X', e_{A'}].$$

Then we have $\theta(\theta'[X'])=\theta[X', e_{A'}]=[X']\circ e_{A'}=[X']$, and $\theta'(\theta[X, \alpha])=\theta'[X\bigcirc\alpha]=[X\bigcirc\alpha, e_{A'}]=[X, \alpha]$, for the map $X\bigcirc\alpha\to X$ paired with the commutative diagram

$$\begin{array}{rccc} e_{A'}: & A' & \Rightarrow & A' \\ & \Vert & & \downarrow{\scriptstyle\alpha} \\ \alpha: & A' & \to & A \end{array}$$

gives the relation $(X\bigcirc\alpha, e_{A'})\simeq(X, \alpha)$. This proves that through θ and its inverse θ' we may identify $\overline{S\ast E^0}(A', B)$ with $\bar{S}(A', B)$. Since for the product (3.5.4) we have $\theta([X]\circ[\alpha])=\theta[X, \alpha]=[X]\circ\alpha$, the product (3.5.4) coincides with the product (Proposition 3.3)

$$\circ: \ \bar{S}(A, B)\times\mathrm{Hom}(A', A)\to\bar{S}(A', B).$$

In this sense $[X]\circ\alpha$ is a special case of the composition product (3.5.2). In particular the usual composition

$$\circ: \ \mathrm{Hom}(A', A)\times\mathrm{Hom}(A'', A')\to\mathrm{Hom}(A', A)$$

is also a special case of (3.5.2). Dually if translations always exist in S, then the composition product for S and the span E^0: $\mathcal{E}^0\mathcal{B}\to\mathcal{B}\times\mathcal{B}$ reduces to the product (Proposition 3.3*)

$$\circ: \ \mathrm{Hom}(B, B')\times\bar{S}(A, B)\to\bar{S}(A, B').$$

Thus when S, T are regular spans, the general associativity formula (3.5.3) leads to the following formulas on the composition product (3.5.2):

Naturality: $[Y]\circ([X]\circ\alpha)=([Y]\circ[X])\circ\alpha$; $(\gamma\circ[Y])\circ[X]=\gamma\circ([Y]\circ[X])$

$$([X]\in\bar{S}(A, B), \ [Y]\in\bar{T}(B, C), \ \alpha: A'\to A, \ \gamma: C\to C').$$

Balancedness: $[Y']\circ(\beta\circ[X])=([Y']\circ\beta)\circ[X]$

$$([X]\in\bar{S}(A, B), \ [Y']\in\bar{T}(B', C), \ \beta: B\to B').$$

These formulas can also be verified directly with little defficulty.

3.6. We now study the additive case. For an additive category $\mathcal{A}$ the regular span E^0: $\mathcal{E}^0\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ is additive, and addition in $\bar{E}^0(A, B)$ defined by (3.3.1) as $[\varphi_1]+[\varphi_2]=\nabla_B\circ[\varphi_1\oplus\varphi_2]\circ\Delta_A$ coincides with the addition in $\mathrm{Hom}(A, B)$. In other

words the identification $\bar{E}^0(A, B)=\mathrm{Hom}(A, B)$ respects addition. Let $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$, $T: \mathcal{Y}\to\mathcal{B}\times\mathcal{C}$ be regular additive spans. As was shown earlier the direct sum in $\mathcal{X}$ and in $\mathcal{Y}$ can be and shall be so chosen that they cover preassigned direct sums in $\mathcal{A}$, in $\mathcal{B}$, and in $\mathcal{C}$. Namely for $X_\nu\in S^{-1}(A_\nu, B_\nu)$, $Y_\nu\in T^{-1}(B_\nu, C_\nu)$ $(\nu=1, 2)$ and direct sums $A_1\oplus A_2$, $B_1\oplus B_2$, $C_1\oplus C_2$, we have $X_1\oplus X_2\in S^{-1}(A_1\oplus A_2, B_1\oplus B_2)$, $Y_1\oplus Y_2\in T^{-1}(B_1\oplus B_2, C_1\oplus C_2)$, and so $(Y_1\oplus Y_2, X_1\oplus X_2)$ is an object in $(T\ast S)^{-1}(A_1\oplus A_2, C_1\oplus C_2)$, which is the direct sum of (Y_1, X_1) and (Y_2, X_2). The subcategory $\mathcal{Y}\ast\mathcal{X}$ of the additive category $\mathcal{Y}\times\mathcal{X}$, being closed under addition of maps, is thus an additive category. Therefore the composed span $T\ast S: \mathcal{Y}\ast\mathcal{X}\to\mathcal{A}\times\mathcal{C}$ is additive regular, and $\overline{T\ast S}(A, C)$ is an additive group. We now show that for regular additive spans $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$. $T: \mathcal{Y}\to\mathcal{B}\times\mathcal{C}$ the composition product (3.5.2) is bilinear. Namely we have

Bilinearity:
$$[Y]\circ([X_1]+[X_2])=[Y]\circ[X_1]+[Y]\circ[X_2],$$
$$([Y_1]+[Y_2])\circ[X]=[Y_1]\circ[X]+[Y_2]\circ[X],$$
$$([X], [X_1], [X_2]\in\bar{S}(A, B)\,;\,[Y], [Y_1], [Y_2]\in\bar{T}(B, C)).$$

In fact we have
$$\begin{aligned}[Y]\circ([X_1]+[X_2])&=[Y]\circ\nabla_B\circ[X_1\oplus X_2]\circ\Delta_A=\nabla_C\circ[Y\oplus Y]\circ[X_1\oplus X_2]\circ\Delta_A\\&=\nabla_C\circ[Y\oplus Y, X_1\oplus X_2]\circ\Delta_A=\nabla_C\circ[(Y, X_1)\oplus(Y, X_2)]\circ\Delta_A\\&=[Y, X_1]+[Y, X_2]=[Y]\circ[X_1]+[Y]\circ[X_2],\end{aligned}$$

and the second bilinearity follows by duality. Summarizing we have:

COMPOSITION PRODUCT THEOREM I. *For regular additive spans $S: \mathcal{X}\to\mathcal{A}\times\mathcal{B}$, $T: \mathcal{Y}\to\mathcal{B}\times\mathcal{C}$ the composition product*
$$\circ:\ \bar{T}(B, C)\times\bar{S}(A, B)\to\overline{T\ast S}(A, C)$$
is natural, balanced, and bilinear.

COROLLARY. *For a regular* S-*category* $(\mathcal{A}, \mathcal{S})$ *the composition product*
$$\circ:\ \mathrm{Ext}^q{}_{(\mathcal{A}, \mathcal{S})}(B, C)\times\mathrm{Ext}^p{}_{(\mathcal{A}, \mathcal{S})}(A, B)\to\mathrm{Ext}^{p+q}{}_{(\mathcal{A}, \mathcal{S})}(A, C)$$
is associative, natural, balanced, and bilinear.

We set $\mathrm{Ext}_{(\mathcal{A}, \mathcal{S})}(A, B)=\mathrm{Hom}(A, B)\oplus\sum_{n=1}^{\infty}\mathrm{Ext}^n{}_{(\mathcal{A}, \mathcal{S})}(A, B)$. Then the conclusion of the corollary may be restated as follows: *With* $\mathrm{Hom}(A, B)$ *replaced by* $\mathrm{Ext}_{(\mathcal{A}, \mathcal{S})}(A, B)$ *and with composition given by the composition product, the additive categyry $\mathcal{A}$ is extended to an additive category $\mathcal{S}$-*Ext *$\mathcal{A}$ having the same objects as $\mathcal{A}$.*

§ 4. Tensor product and satellites of functors.

4.0. Let $\mathcal{C}$ be a category. By a (*left*) *$\mathcal{C}$-group* we mean a covariant functor M of $\mathcal{C}$ with values in the category $\mathcal{M}$ of additive groups and homomorphisms. For

a map $\gamma: C \to C'$ in $\mathcal{C}$ and for an element $m \in M(C)$ we denote by $\gamma \circ m$ the element $M(\gamma)m \in M(C')$. Also by a $\mathcal{C}^*$*-group* (or a *right* $\mathcal{C}$*-group*) we mean a contravariant functor $K: \mathcal{C} \to \mathfrak{A}$, and for $\gamma: C' \to C$, $k \in K(C)$ we denote $M(\gamma)k \in K(C')$ by $k \circ \gamma$. Functors of several variables with values in $\mathfrak{A}$ will accordingly be called $\mathfrak{B}$-$\mathcal{C}$-groups, $\mathfrak{B}^*$-$\mathcal{C}$-groups, etc.

Let H be a $\mathcal{C}^*$-$\mathcal{C}$-group, and G an additive group. By a *balanced homomorphism* $\mu: G \to H$ we mean a system of homomorphisms $\mu(C): G \to H(C, C)$ defined for all objects C in $\mathcal{C}$ such that for every map $\gamma: C \to C'$ in $\mathcal{C}$ commutativity holds in the diagram

$$\begin{array}{ccc} G & \xrightarrow{\mu(C)} & H(C, C) \\ {\scriptstyle \mu(C')}\downarrow & & \downarrow{\scriptstyle \gamma\circ} \\ H(C', C') & \xrightarrow{\circ\gamma} & H(C, C'). \end{array}$$

Also by a *balanced homomorphism* $\lambda: H \to G$ we mean a system of homomorphisms $\lambda(C): H(C, C) \to G$ defined for all objects C in $\mathcal{C}$ such that for every map $\gamma: C \to C'$ in $\mathcal{C}$ commutativity holds in the diagram

$$\begin{array}{ccc} H(C', C) & \xrightarrow{\gamma\circ} & H(C', C') \\ \downarrow{\scriptstyle \circ\gamma} & & \downarrow{\scriptstyle \lambda(C')} \\ H(C, C) & \xrightarrow{\lambda(C)} & G. \end{array}$$

An additive group I together with a balanced homomorphism $\mathcal{C}: H \to I$ is called *integration* of $\mathcal{C}^*$-$\mathcal{C}$-group H if it is universal among balanced homomorphisms from H, i.e., if for any balanced homomorphism $\lambda: H \to G$ there is a unique homomorphism $\zeta: I \to G$ such that $\zeta \circ \theta(C) = \lambda(C)$ for every object C in $\mathcal{C}$. Similarly an additive group J together with a balanced homomorphism $\tau: J \to H$ is called *cointegration* of H if for any balanced homomorphism $\mu: G \to H$ there is a unique homomorphism $\zeta: G \to J$ such that $\tau(C) \circ \zeta = \mu(C)$ for every object C in $\mathcal{C}$.

Clearly the integration and cointegration of H is unique upto equivalence. When the totality of maps in $\mathcal{C}$ is a set, they are legitimately given as follows: For a map $\gamma: C \to C'$ in $\mathcal{C}$ we put $H(\gamma) = H(C', C)$, $H(\gamma^*) = H(C, C')$, and define homomorphisms

$$\partial_\gamma: \quad H(\gamma) \to H(C, C) \oplus H(C', C'),$$
$$\delta_\gamma: \quad H(C, C) \oplus H(C', C') \to H(\gamma^*),$$

by

$$\partial_\gamma(h') = h' \circ \gamma \oplus (-\gamma \circ h') \qquad (h' \in H(\gamma)),$$
$$\delta_\gamma(h \oplus h'') = \gamma \circ h - h'' \circ \gamma \qquad (h \in H(C, C),\ h'' H(C', C')).$$

Denote by Σ_0 and Π^0 the direct sum $\sum H(C, C)$ and the direct product $\Pi H(C, C)$ (C running over all objects in $\mathcal{C}$) respectively. Also denote by Σ_1 and Π^1 the direct sum $\sum H(\gamma)$ and the direct product $\Pi H(\gamma^*)$ (γ running over all maps in $\mathcal{C}$)

respectively. Then ∂_γ, δ_γ are extended to homomorphisms

$$\partial:\ \Sigma_1\to\Sigma_0 \qquad \delta:\ \Pi^0\to\Pi^1,$$

where if $C=C'$, ∂_γ should be followed by the codiagonal map $H(C,C)\oplus H(C,C)\to H(C,C)$ and δ_γ should be preceded by the diagonal map $H(C,C)\to H(C,C)\oplus H(C,C)$. We set $I=\operatorname{Coker}\partial$, $J=\operatorname{Ker}\delta$, and define $\theta(C): H(C,C)\to I$ as the canonical injection $H(C,C)\to\Sigma_0$ composed with the natural projection $\Sigma_0\to\operatorname{Coker}\partial$, $\tau(C): J\to H(C,C)$ as the natural injection $J\to\Pi^0$ composed with the canonical projection $\Pi^0\to H(C,C)$. Then $\theta: H\to I$, $\tau: J\to H$ give the integration and cointegration of H respectively. We shall write $\int_{\mathcal{C}} H$ for the integration I of H, and $\int_{\mathcal{C}}^{*} H$ for the cointegration J of H. When the totality of maps in $\mathcal{C}$ is not a set, we have to impose certain conditions for a legitimate existence of integration and cointegration. We shall put aside those metamathematical preoccupations. The reader should thus understand that, in this paper, an additive group may have the underlying collection which is not a set. The above explicit presentation of $\int_{\mathcal{C}} H$ and $\int_{\mathcal{C}}^{*} H$ may then be considered as valid regardless of whether the totality of maps in $\mathcal{C}$ is a set or not.

By merely interchanging the variables we also define the integration and cointegration of a $\mathcal{C}$-$\mathcal{C}^*$-group. In dealing with functors of more variables, we shall often inscribe x (or y, z) to indicate the two entries to be considered in the (co)integration, namely we write $\int_{x\in\mathcal{C}} H(\cdots,x,\cdots,x,\cdots)$. This is based on the following fact: Let H, H' be $\mathcal{C}$-$\mathcal{C}^*$-groups, and let $\theta: H\to\int_{\mathcal{C}} H$, $\theta': H'\to\int_{\mathcal{C}} H'$ be the integrations. Then a natural transformation $\eta: H\to H'$ induces a unique homomorphism $\int_{\mathcal{C}}\eta: \int_{\mathcal{C}} H\to\int_{\mathcal{C}} H'$ such that $\left(\int_{\mathcal{C}}\eta\right)\circ\theta(C)=\theta'(C)\circ\eta(C,C)$. Thus if H is a $\mathcal{B}$-$\mathcal{C}^*$-$\mathcal{C}$-group, then $\int_{x\in\mathcal{C}}^{(*)} H(b,x,x)$ is a $\mathcal{B}$-group. On this account, for an $\mathcal{A}$-$\mathcal{B}^*$-$\mathcal{B}$-$\mathcal{C}^*$-$\mathcal{C}$ group H we have

$$\int_{y\in\mathcal{B}}\int_{x\in\mathcal{C}} H(a,y,y,x,x)=\int_{x\in\mathcal{C}}\int_{y\in\mathcal{B}} H(a,y,y,x,x) \tag{4.0.1}$$

Thereby the $\int$'s can be replaced by $\int^*$. Next for a $\mathcal{B}$-group M and a $\mathcal{C}$-group N, $M\otimes N=N(b)\otimes N(c)$ is a $\mathcal{B}$-$\mathcal{C}$-group, and $\operatorname{Hom}(M,N)=\operatorname{Hom}(M(b),N(c))$ is a $\mathcal{B}^*$-$\mathcal{C}$-group. For an $\mathcal{A}$-group M and a $\mathcal{B}$-$\mathcal{C}$-$\mathcal{C}^*$-group H we have:

$$\int_{x\in\mathcal{C}} M(a)\otimes H(b,x,x)=M(a)\otimes\int_{x\in\mathcal{C}} H(b,x,x), \tag{4.0.2}$$

$$\int_{x\in\mathcal{C}}^{*} \operatorname{Hom}(M(a),H(b,x,x))=\operatorname{Hom}\left(M(a),\int_{x\in\mathcal{C}}^{*} H(b,x,x)\right), \tag{4.0.3}$$

$$\int^{*}_{x\in C}\operatorname{Hom}(H(b,x,x),M(a))=\operatorname{Hom}\Big(\int_{x\in C}H(b,x,x),M(a)\Big). \tag{4.0.4}$$

Verifications of these formulas are all straightforward, and so will be omitted.

Let M, M' be C-groups and let $\sigma_1, \sigma_2: M'\to M$ be natural tranformations. We set

$$(\sigma_1+\sigma_2)(C)m=\sigma_1(C)m+\sigma_2(C)m \qquad (m\in M(C)).$$

Then $(\sigma_1+\sigma_2)(C)$ is a homomorphism $M(C)\to M'(C)$. Since $\gamma\circ: M'(C)\to M'(C')$, $(\gamma: C\to C')$ is a homomorphism, we have $\gamma\circ((\sigma_1+\sigma_2)(C)m)=\gamma\circ(\sigma_1(C)m)+\gamma\circ(\sigma_2(C)m)$ $=\sigma_1(C')(\gamma\circ m)+\sigma_2(C')(\gamma\circ m)=(\sigma_1+\sigma_2)(C')(\gamma\circ m)$, and so $\sigma_1+\sigma_2$ is a natural transformation $M\to M'$. For natural transformations $\eta_1, \eta_2: H\to H'$ of C-C^*-groups H, H', and for the integrations $\theta: H\to\int_C H, \theta': H'\to\int_C H'$, we have

$$\Big(\int_C\eta_1+\int_C\eta_2\Big)\circ\theta(C)=\Big(\int_C\eta_1\Big)\circ\theta(C)+\Big(\int_C\eta_2\Big)\circ\theta(C)=\theta'(C)\circ(\eta_1(C,C)+\eta_2(C,C)).$$

Therefore we have $\int_C(\eta_1+\eta_2)=\int_C\eta_1+\int_C\eta_2$. In this sense (co)integration is additive.

4.1. Let M, M' be C-groups. Then $\operatorname{Hom}(M, M')=\operatorname{Hom}(M(c), M'(c'))$ is a C^*-C-group. We denote by $\operatorname{Hom}_C(M, M')$ the additive group of natural transformations $M\to M'$, which we call *C-homomorphisms*. Let σ be a C-homomorphism $M\to M'$. Then by definition σ assigns to each object C in C a homomorphism $\sigma(C): M(C)$ $\to M'(C)$ such that for any map $\gamma: C\to C'$ we have $\gamma\circ(\sigma(C)m)=\sigma(C')(\gamma\circ m)$ $(m\in M(C))$. We now define a system of homomorphisms $\tau(C):\operatorname{Hom}_C(M, M')\to\operatorname{Hom}(M(C), M'(C))$ by $\tau(C)\sigma=\sigma(C)$. Then we have $(((\gamma\circ)\circ\tau(C))\sigma)m=(\gamma\circ(\tau(C)\sigma))m=\gamma\circ(\sigma(C)m)=\sigma(C')(\gamma\circ$ $m)=(\sigma(C')\circ\gamma)m=((\tau(C')\circ(\circ\gamma))\sigma)m$ $(\sigma\in\operatorname{Hom}_C(M, M'),\ m\in M(C))$, and so τ is a balanced homomorphism $\operatorname{Hom}_C(M, M')\to\operatorname{Hom}(M, M')$.

Let $\mu: G\to\operatorname{Hom}(M, M')$ be a balanced homomorphism. For an element $g\in G$ we put $\sigma_g(C)=\mu(C)g\in\operatorname{Hom}(M(C), M'(C))$. Then σ_g is a C- homomorphism $M\to M'$, and putting $\zeta(g)=\sigma_g$ we obtain a homomorphism $\zeta: G\to\operatorname{Hom}_C(M, M')$. This is clearly the unique homomorphism such that $\tau(C)\circ\zeta=\mu(C)$, and so $\tau:\operatorname{Hom}_C(M, M')$ $\to\operatorname{Hom}(M, M')$ is the cointegration of the C^*-C-group $\operatorname{Hom}(M, M')$. Namely we have

$$\int^{*}_{x\in C}\operatorname{Hom}(M(x), M'(x))=\operatorname{Hom}_C(M, M'). \tag{4.1.1}$$

This formula can be reviewed also as follows: Any system σ of homomorphisms $\sigma(C): M(C)\to M'(C)$ can be considered as an element in the direct product Π^0 $=\Pi\operatorname{Hom}(M(C), M'(C))$. Then σ is a C-homomorphism if and only if $\delta\sigma=0$, and so we get $\operatorname{Hom}_C(M, M')=\operatorname{Ker}\delta$. Note that (4.1.1) holds also for C^*-groups M, M', in

which case $\mathrm{Hom}(M, M')$ is a $\mathcal{C}$-$\mathcal{C}^*$-group. If M is an $\mathcal{A}$-$\mathcal{C}$-group, and if M' is a $\mathcal{C}^*$-$\mathcal{B}$-group then $\mathrm{Hom}_{\mathcal{C}}(M(a, x), M'(x, b))$ is an $\mathcal{A}^*$-$\mathcal{B}$-group.

We now consider $\otimes$, the adjoint operation of Hom. Let M be a $\mathcal{C}^*$-group, and M' a $\mathcal{C}$-group. Then $M\otimes M'$ being a $\mathcal{C}^*$-$\mathcal{C}$-group we define the *tensor product* of M and M' *over* $\mathcal{C}$ by

$$M\otimes_{\mathcal{C}}M'=\int_{x\in\mathcal{C}} M(x)\otimes M'(x).$$

If M is an $\mathcal{A}$-$\mathcal{C}^*$-group, and if M' is a $\mathcal{C}$-$\mathcal{B}$-group, then $M(a, x)\otimes_{\mathcal{C}}M'(x, b)$ is an $\mathcal{A}$-$\mathcal{B}$-group. Let M be an $\mathcal{A}$-$\mathcal{B}^*$-group, M' a $\mathcal{B}$-$\mathcal{C}^*$-group, and M'' a $\mathcal{C}$-$\mathcal{D}$-group. Following [4] this situation will be designated as $({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{B}}M'_{\mathcal{C}}, {}_{\mathcal{C}}M''_{\mathcal{D}})$. Obviously the $\mathcal{A}$-$\mathcal{B}^*$-$\mathcal{B}$-$\mathcal{C}^*$-$\mathcal{C}$-$\mathcal{D}$-groups $M\otimes(M'\otimes M'')$ and $M\otimes(M'\otimes M'')$ can be identified with each other. From this we get the associativity:

(4.1.2) $$(M\otimes_{\mathcal{B}}M')\otimes_{\mathcal{C}}M''=M\otimes_{\mathcal{B}}(M'\otimes_{\mathcal{C}}M''),\ ({}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{B}}M'_{\mathcal{C}}, {}_{\mathcal{C}}M''_{\mathcal{D}}).$$

In fact we have

$$\begin{aligned}
M(a, x)\otimes_{\mathcal{B}}(M'(x, y)\otimes_{\mathcal{C}}M''(y, d))&=M(a, x)\otimes_{\mathcal{B}}\int_{y\in\mathcal{C}} M'(x, y)\otimes M''(y, d)\\
&=\int_{x\in\mathcal{B}}\Big(M(a, x)\otimes\int_{y\in\mathcal{C}} M'(x, y)\otimes M''(y, d)\Big)\\
&=\int_{x\in\mathcal{B}}\int_{y\in\mathcal{C}} M(a, x)\otimes M'(x, y)\otimes M''(y, d) && (4.0.2)\\
&=\int_{y\in\mathcal{C}}\int_{x\in\mathcal{B}} M(a, x)\otimes M'(x, y)\otimes M''(y, d) && (4.0.1)\\
&=\int_{y\in\mathcal{C}}\left(\left(\int_{x\in\mathcal{B}} M(a, x)\otimes M'(x, y)\right)\otimes M''(y, d)\right) && (4.0.2)\\
&=(M(a, x)\otimes_{\mathcal{B}}M'(x, y))\otimes_{\mathcal{C}}(y, d).
\end{aligned}$$

Similarly using (4.0.1), (4.0.3), (4.0.4) we get the following formulas:

(4.1.3) $$\mathrm{Hom}_{\mathcal{B}}(M\otimes M', M'')=\mathrm{Hom}_{\mathcal{C}}(M, \mathrm{Hom}_{\mathcal{B}}(M', M''),\quad ({}_{\mathcal{B}}M_{\mathcal{C}}, {}_{\mathcal{C}}M'_{\mathcal{A}}, {}_{\mathcal{B}}M''_{\mathcal{D}}).$$

(4.1.4) $$\mathrm{Hom}_{\mathcal{B}}(M\otimes_{\mathcal{C}}M', M'')=\mathrm{Hom}_{\mathcal{C}}(M', \mathrm{Hom}_{\mathcal{B}}(M, M'')),\quad ({}_{\mathcal{A}}M_{\mathcal{C}}, {}_{\mathcal{C}}M'_{\mathcal{B}}, {}_{\mathcal{D}}M''_{\mathcal{B}}).$$

As was remarked in §4.0, integration is realized as $\mathrm{Coker}(\partial: \Sigma_1\to\Sigma_0)$. This now means that in the situation $(M_{\mathcal{C}}, {}_{\mathcal{C}}M')$ the tensor product $M\otimes_{\mathcal{C}}M'$ is the factor group $\Sigma_0/\bar{R}_{\mathcal{C}}$, where Σ_0 is the direct sum $\sum M(C)\otimes M'(C)$ and $\bar{R}_{\mathcal{C}}$ is the subgroup of Σ_0 generated by elements of the form $m\circ\gamma\otimes m'-m\otimes\gamma\circ m'$ ($m\in M(C')$, $m'\in M'(C)$, $\gamma: C\to C'$). Thus $M\otimes_{\mathcal{C}}M'$ is also represented as the factor group $F/R_1\cup R_2\cup R_{\mathcal{C}}$, where F is the free additive group generated by the pairs

$$(m, m') \qquad (m \in M(C),\ m' \in M'(C),\ C \in \mathcal{C}),$$

R_1 is the subgroup generated by elements of the form

$$(m_1+m_2, m')-(m_1, m')-(m_2, m'),$$

R_2 is generated by

$$(m, m_1'+m_2')-(m, m_1')-(m, m_2'),$$

and R_C is generated by

$$(m\circ\gamma, m')-(m, \gamma\circ m').$$

Remark. A monoid π can be considered as a category with only one object. Then π-groups and π^*-groups mean the usual left π-groups and right π-groups. The tensor product over the category π coincides with the tensor product over the monoid π. Thus the range category $\mathcal{M}$ can easily be replaced by the categories ${}_\Lambda\mathcal{M}$, $\mathcal{M}_\Lambda$, etc. The full subcategory Λ^∞ of $\mathcal{M}_\Lambda$ consisting of the objects 0, Λ, $\Lambda\oplus\Lambda$, $\Lambda\oplus\Lambda\oplus\Lambda,\cdots$ is an additive category. Left (right) Λ-modules are then interpreted as left (right) Λ^∞-modules, i.e. additive covariant (contravariant) functors $\Lambda^\infty\to\mathcal{M}$. Here again the tensor product over the category Λ^∞ coincides with the tensor product over the ring Λ with unit. Moreover additive functors $\mathcal{A}\to{}_\Lambda\mathcal{M}$ are interpreted as $\mathcal{A}$-Λ^∞-modules (or $\mathcal{A}^*$-Λ^∞-modules), and in this way our range category $\mathcal{M}$ can readily be replaced by ${}_\Lambda\mathcal{M}$.

4.2. We now suppose $\mathcal{A}, \mathcal{B}, \mathcal{C},\cdots$ are additive categories, and restrict our attention to $\mathcal{A}$-modules (i. e. additive covariant functors $\mathcal{A}\to\mathcal{M}$), $\mathcal{B}$-$\mathcal{C}^*$-$\mathcal{C}$-modules, etc. For an $\mathcal{A}$-$\mathcal{B}^*$-module M and a $\mathcal{B}$-$\mathcal{C}^*$-module M', $M(a, x)\otimes_{\mathcal{B}} M'(x, c)$ is primarily an $\mathcal{A}$-$\mathcal{C}^*$-group. However, since $M\otimes M'$ is an $\mathcal{A}$-$\mathcal{B}^*$-$\mathcal{B}$-$\mathcal{C}$-module, and since integration is additive, $M\otimes_{\mathcal{B}} M'$ is actually an $\mathcal{A}$-$\mathcal{C}^*$-module. Similarly for an $\mathcal{A}$-$\mathcal{B}$-module M and a $\mathcal{B}$-$\mathcal{C}$-module M', $\mathrm{Hom}_{\mathcal{B}}(M, M')$ is an $\mathcal{A}^*$-$\mathcal{C}$-module.

PROPOSITION 4.1. *Let M, M' be $\mathcal{A}$-modules and let $\sigma(A): M(A)\to M'(A)$ be a system of mappings (not necessarily homomorphisms) such that for every map $\alpha: A\to A'$ we have the naturality $\alpha\circ(\sigma(A)m)=\sigma(A')(\alpha\circ m)$ $(m\in M(A))$. Then the mappings $\sigma(A)$ are necessarily homomorphisms, and so σ is an $\mathcal{A}$-homomorphism $M\to M'$.*

PROOF Let m_1, m_2 be elements in $M(A)$. Take the direct sum $(\alpha_1, \alpha_2, \bar\alpha_1, \bar\alpha_2)$: $A^2\approx A\oplus A$, and set $\nabla_A=\bar\alpha_1+\bar\alpha_2$. Then we have

$$m_1+m_2=\nabla_A\circ\alpha_1\circ m_1+\nabla_A\circ\alpha_2\circ m_2=\nabla_A\circ(\alpha_1\circ m_1+\alpha_2\circ m_2),$$

and so we obtain

$$\begin{aligned}\sigma(A)(m_1+m_2)&=\nabla_A\circ\sigma(A^2)(\alpha_1\circ m_1+\alpha_2\circ m_2)\\&=\bar\alpha_1\circ\sigma(A^2)(\alpha_1\circ m_1+\alpha_2\circ m_2)+\bar\alpha_2\circ\sigma(A^2)(\alpha_1\circ m_1+\alpha_2\circ m_2)\end{aligned}$$

$$=\sigma(A)(\bar{\alpha}_1\circ\alpha_1\circ m_1+\bar{\alpha}_1\circ\alpha_2\circ m_2)+\sigma(A)(\bar{\alpha}_2\circ\alpha_1\circ m_1+\bar{\alpha}_2\circ\alpha_2\circ m_2)$$
$$=\sigma(A)(m_1+0)+\sigma(A)(0+m_2)$$
$$=\sigma(A)m_1+\sigma(A)m_2.$$

PROPOSITION 4.2. *Let M be an $\mathcal{A}^*$-module, and M' an $\mathcal{A}$-module. With regard to the expression $M\otimes_{\mathcal{A}}M'=F/R_1\cup R_2\cup R_{\mathcal{A}}$ as given in §4.1, let $R_\oplus$ be the subgroup of F generated by elements of the form*

$$(m_1, m_1')+(m_2, m_2')-(m_1\circ\bar{\alpha}_1+m_2\circ\bar{\alpha}_2, \alpha_1\circ m_1'+\alpha_2\circ m_2'),$$

where $m_\nu\in M(A_\nu)$, $m_\nu'\in M'(A_\nu)$ $(\nu=1, 2)$ and $(\alpha_1, \alpha_2, \bar{\alpha}_1, \bar{\alpha}_2): A^2\approx A_1\oplus A_2$. Then we have

$$R_1\cup R_2\cup R_{\mathcal{A}}=R_\oplus\cup R_{\mathcal{A}}=R_1\cup R_{\mathcal{A}}=R_2\cup R_{\mathcal{A}}.$$

PROOF It will suffice to verify the inclusions (i) $R_\oplus\subset R_1\cup R_{\mathcal{A}}$, (ii) $R_2\subset R_\oplus\cup R_{\mathcal{A}}$.

Ad (i):
$$\begin{aligned}&(m_1\circ\bar{\alpha}_1+m_2\circ\bar{\alpha}_2, \alpha_1\circ m_1'+\alpha_2\circ m_2')\\ &\equiv(m_1\circ\bar{\alpha}_1, \alpha_1\circ m_1'+\alpha_2\circ m_2')+(m_2\circ\bar{\alpha}_2, \alpha_1\circ m_1'+\alpha_2\circ m_2') &&\pmod{R_1}\\ &\equiv(m_1, \bar{\alpha}_1\circ\alpha_1\circ m_1'+\bar{\alpha}_2\circ\alpha_1\circ m_2')+(m_2, \bar{\alpha}_2\circ\alpha_1\circ m_1'+\bar{\alpha}_2\circ\alpha_2\circ m_2') &&\pmod{R_{\mathcal{A}}}\\ &=(m_1, m_1'+0)+(m_2, 0+m_2')=(m_1, m_1')+(m_2, m_2').\end{aligned}$$

Ad (ii): We first note the identity $m_1'+m_2'=\nabla_A\circ(\alpha_1\circ m_1'+\alpha_2\circ m_2')$ $(A_1=A_2=A$, $\nabla_A=\bar{\alpha}_1+\bar{\alpha}_2$, $m_1', m_2'\in M'(A))$. Then we get

$$\begin{aligned}(m, m_1'+m_2')&\equiv(m\circ\nabla_A, \alpha_1\circ m_1'+\alpha_2\circ m_2') &&\pmod{R_{\mathcal{A}}}\\ &=(m\circ\bar{\alpha}_1+m\circ\bar{\alpha}_2, \alpha_1\circ m_1'+\alpha_2\circ m_2')\\ &\equiv(m, m_1')+(m, m_2') &&\pmod{R_\oplus}.\end{aligned}$$

COROLLARY. *Every element in $M\otimes_{\mathcal{A}}M'=F/R_1\cup R_2\cup R_{\mathcal{A}}$ can be represented by one of the generators (m, m') $(m\in M(A), m'\in M'(A))$ of F.*
In fact we have

$$\begin{aligned}(m_1, m_1')+(m_2, m_2')&\equiv(m_1\circ\bar{\alpha}_1+m_2\circ\bar{\alpha}_2, \alpha_1\circ m_1'+\alpha_2\circ m_2') &&\pmod{R_\oplus},\\ 0\quad&\equiv(O_{M(A)}, m') &&\pmod{R_1}\\ &\qquad(O_{M(A)}=\text{zero element}\in M(A),\ m'\in M'(A)),\\ -(m, m')&\equiv(-m, m') &&\pmod{R_1}.\end{aligned}$$

We shall denote by $m\otimes_{\mathcal{A}}m'$ the element of $M\otimes_{\mathcal{A}}M'=F/R_1\cup R_2\cup R_{\mathcal{A}}$ represented by (m, m').

Let K be an $\mathcal{A}^*$-$\mathcal{B}$-module. We denote by $K(\mathcal{A}, \mathcal{B})$ the preadditive category given by the following data:

(i) An object in $K(\mathcal{A}, \mathcal{B})$ means a triple $\langle A, B, k\rangle$ of an object A in $\mathcal{A}$, an object B in $\mathcal{B}$, and an element $k\in K(A, B)$.

(ii) A map $\langle A, B, k\rangle\to\langle A', B', k'\rangle$ in $K(\mathcal{A}, \mathcal{B})$ means a quadruple $\langle\alpha, \beta, k, k'\rangle$

of a map $\alpha: A \to A'$, a map $\beta: B \to B'$, and elements $k \in K(A, B)$, $k' \in K(A', B')$ such that $\beta \circ k = k' \circ \alpha$.

(iii) Compositioh of maps is given by

$$\langle \alpha', \beta', k', k'' \rangle \circ \langle \alpha, \beta, k, k' \rangle = \langle \alpha' \circ \alpha, \beta' \circ \beta, k', k'' \rangle.$$

(iv) Addition of maps $\langle A, B, k \rangle \to \langle A', B', k' \rangle$ is given by

$$\langle \alpha_1, \beta_1, k, k' \rangle + \langle \alpha_2, \beta_2, k, k' \rangle = \langle \alpha_1 + \alpha_2, \beta_1 + \beta_2, k, k' \rangle.$$

In this category $\langle \emptyset, \emptyset, 0 \rangle$ ($\{0\} = K(\emptyset, \emptyset)$) is the neutral object. Given two objects $\langle A_\nu, B_\nu, k_\nu \rangle$ ($\nu = 1, 2$) in $K(\mathcal{A}, \mathcal{B})$, we take the direct sums $(\alpha_1, \alpha_2, \bar{\alpha}_1, \bar{\alpha}_2): A \approx A_1 \oplus A_2$, $(\beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2): B \approx B_1 \oplus B_2$, and put $k = \beta_1 \circ k_1 \circ \bar{\alpha}_1 + \beta_2 \circ k_2 \circ \bar{\alpha}_2 \in K(A, B)$. Then $\langle A, B, k \rangle$ is an object in $K(\mathcal{A}, \mathcal{B})$, and we have

$$(\langle \alpha_1, \beta_1, k_1, k \rangle, \langle \alpha_2, \beta_2, k_2, k \rangle, \langle \bar{\alpha}_1, \bar{\beta}_1, k, k_1 \rangle, \langle \bar{\alpha}_2, \bar{\beta}_2, k, k_2 \rangle):$$
$$\langle A, B, k \rangle \approx \langle A_1, B_1, k_1 \rangle \oplus \langle A_2, B_2, k_2 \rangle.$$

In this way $K(\mathcal{A}, \mathcal{B})$ becomes an additive category. We define a covariant functor $\langle K \rangle$: $K(\mathcal{A}, \mathcal{B}) \to \mathcal{A} \times \mathcal{B}$ by $\langle K \rangle \langle A, B, k \rangle = (A, B)$, $\langle K \rangle \langle \alpha, \beta, k, k' \rangle = (\alpha, \beta)$.

The additive span $\langle K \rangle$ thus obtained is regular, for the cotranslation of $\langle A, B, k \rangle$ by $\alpha: A' \to A$ and the translation of $\langle A, B, k \rangle$ by $\beta: B \to B'$ are given respectively by

$$\langle \alpha, e_B, k \circ \alpha, k \rangle : \langle A', B, k \circ \alpha \rangle \to \langle A, B, k \rangle,$$
$$\langle e_A, \beta, k, \beta \circ k \rangle : \langle A, B, k \rangle \to \langle A, B', \beta \circ k \rangle.$$

In this span the similarity class of $\langle A, B, k \rangle$ consists of $\langle A, B, k \rangle$ alone, and the $\mathcal{A}^*$-$\mathcal{B}$-module $\overline{\langle K \rangle}(a, b)$ derived from this span is identical with the $\mathcal{A}^*$-$\mathcal{B}$-module K. For the $\mathcal{A}^*$-$\mathcal{A}$-module Hom of $\mathcal{A}$ the associated span $\langle \mathrm{Hom} \rangle$ is just the span E^0: $\mathcal{E}^0\mathcal{A} \to \mathcal{A} \times \mathcal{A}$ of maps in $\mathcal{A}$.

Let K be an $\mathcal{A}^*$-$\mathcal{B}$-module, and let H be a $\mathcal{B}^*$-$\mathcal{C}$-module. Then the composed span $\langle H \rangle * \langle K \rangle$ being regular and additive, we have the $\mathcal{A}^*$-$\mathcal{C}$-module $\overline{\langle H \rangle * \langle K \rangle}$ together with the composition product.

$$\circ : H(B, C) \times K(A, B) \to \overline{\langle H \rangle * \langle K \rangle}(A, C).$$

Take the tensor product $H(x, C) \otimes_{\mathcal{B}} K(A, x) = F / R_1 \cup R_2 \cup R_{\mathcal{B}}$, and define an epimorphism $\Xi: F \to \overline{\langle H \rangle * \langle K \rangle}(A, C)$ by $\Xi(h, k) = h \circ k = [\langle B, C, h \rangle, \langle A, B, k \rangle]$. Now in the span $\overline{\langle H \rangle * \langle K \rangle}$ over $(\mathcal{A}, \mathcal{C})$ the relation $(\langle B, C, h \rangle, \langle A, B, k \rangle) \rightrightarrows (\langle B', C, h' \rangle, \langle A, B', k' \rangle)$ means that there exists a map $\beta: B \to B'$ such that $h = h' \circ \beta$, $\beta \circ k = k'$. Therefore $\Xi(h, k) = \Xi(h', k')$ implies $(h, k) \equiv (h', k') \pmod{R_{\mathcal{B}}}$. On the other hand, since the composition product is bilinear and balanced, Ξ annihilates the subgroups $R_1, R_2, R_{\mathcal{B}}$. Thus in virtue of the corollary to Proposition 4.2 we get $\operatorname{Ker} \Xi = R_1 \cup$

$R_2 \cup R_{\mathcal{B}}$, and so Ξ induces an isomorphism $H(x, C) \otimes_{\mathcal{B}} K(A, x) \to \overline{\langle H \rangle * \langle K \rangle}(A, C)$ sending $h \otimes_{\mathcal{B}} k$ to $h \circ k$. This isomorphism is natural with respect to the entries A, C, and so we may identify the $\mathcal{A}^*$-$\mathcal{C}$-modules $H(x, c) \otimes_{\mathcal{B}} K(a, x)$, $\overline{\langle H \rangle * \langle K \rangle}(a, c)$.

Let $S: \mathcal{X} \to \mathcal{A} \times \mathcal{B}$, $T: \mathcal{Y} \to \mathcal{B} \times \mathcal{C}$ be regular additive spans. The similarity classification in the composed span $T * S$ can be divided into two steps, first classifying with the middle term B fixed, and then reclassifying with the floating middle term B. Therefore we have $\overline{T * S}(a, c) = \overline{\langle \bar{T} \rangle * \langle \bar{S} \rangle}(a, c)$. This establishes the following:

COMPOSITION PRODUCT THEOREM II. *For regular additive spans* $S: \mathcal{X} \to \mathcal{A} \times \mathcal{B}$, $T: \mathcal{Y} \to \mathcal{B} \times \mathcal{C}$ *we have the natural identification*

$$\overline{T * S} = \bar{T} \otimes_{\mathcal{B}} \bar{S}.$$

Rewriting the composition product as

$$\circ: \bar{T}(B, C) \times \bar{S}(A, B) \to \bar{T} \otimes_{\mathcal{B}} \bar{S}(A, C),$$

we have $[Y] \circ [X] = [Y] \otimes_{\mathcal{B}} [X]$.

The above argument shows also that $(h, k) \equiv (h', k') \pmod{R_1 \cup R_2 \cup R_{\mathcal{B}}}$ implies $(h, k) \equiv (h', k') \pmod{R_{\mathcal{B}}}$. This statement does not contain any material concerning the variables in $\mathcal{A}$ and in $\mathcal{C}$. Actually we have:

PROPOSITION 4.3. *Let $\mathcal{M}$ be an $\mathcal{A}^*$-module, and M' an $\mathcal{A}$-module. With regard to the expression $M \otimes_{\mathcal{A}} M' = F/R_1 \cup R_2 \cup R_{\mathcal{A}}$ we have the congruence $(m_1, m_1') \equiv (m_2, m_2') \bmod R_1 \cup R_2 \cup R_{\mathcal{A}}$ if and only if the congruence holds* $\bmod R_{\mathcal{A}}$.

PROOF. We convert M' to an $\mathcal{M}^*$-$\mathcal{A}$-module $\tilde{M}'(z, a)$ by setting $\tilde{M}'(z, a) = \mathrm{Hom}(z, M'(a))$, and M to an $\mathcal{A}^*$-$\mathcal{M}$-module $\tilde{M}(a, z)$ by setting $\tilde{M}(a, z) = M(a) \otimes z$. Then we have $\tilde{M}(a, Z) = M(a)$, $\tilde{M}'(Z, a) = M'(a)$, and so $\tilde{M}(x) \otimes_{\mathcal{A}} M'(x) = \tilde{M}'(x, Z) \otimes_{\mathcal{A}} \tilde{M}(Z, x)$. Since these tensor products have the same F, R_1, R_2, $R_{\mathcal{A}}$ and since the proposition is true for $\tilde{M}(x, Z) \otimes_{\mathcal{A}} \tilde{M}'(Z, x)$, it is also true for $M(x) \otimes_{\mathcal{A}} M'(x)$, q. e. d.

In virtue of Propositions 4.2, 4.3 the tensor product $M \otimes_{\mathcal{A}} M'$ can be redefined as follows. Consider the pairs (m, m') ($m \in M(A)$, $m' \in M'(A)$, $A \in \mathcal{A}$). For two pairs (m_ν, m_ν') ($m_\nu \in M(A_\nu)$, $m_\nu' \in M'(A_\nu)$, $\nu = 1, 2$) we write $(m_1, m_1') \simeq (m_2, m_2')$ if there exists a map $\alpha: A_1 \to A_2$, such that $m_1 = m_2 \circ \alpha$, $\alpha \circ m_1' = m_2'$. This relation $\simeq$ generates an equivalence relation $\sim$ (similarity) among the pairs (m, m'). We denote the similarity class of (m, m') by $[m, m']$, and define $M \otimes_{\mathcal{A}} M'$ as the totality of similarity classes $[m, m']$. Given two pairs (m_ν, m_ν') ($\nu = 1, 2$) as above, we take the direct sum $(\alpha_1, \alpha_2, \bar{\alpha}_1, \bar{\alpha}_2): A \approx A_1 \oplus A_2$. Then $[m_1 \circ \bar{\alpha}_1 + m_2 \circ \bar{\alpha}_2, \alpha_1 \circ m_1' + \alpha_2 \circ m_2']$ depends only on $[m_1, m_1']$ and $[m_2, m_2']$. Addition in $M \otimes_{\mathcal{A}} M'$ is then defined by

$[m_1, m_1'] + [m_2, m_2'] = [m_1 \circ \tilde{\alpha}_1 + m_2 \circ \tilde{\alpha}_2, \alpha_1 \circ m_1' + \alpha_2 \circ m_2']$. Note that $F/R_{\mathcal{A}}$ is the free additive group generated by $M \otimes_{\mathcal{A}} M'$.

The above proof of Proposition 4.3 shows also that $m \otimes_{\mathcal{A}} m'$ can be treated as though it were the composition product of m and m'. Thus general associativity of composition product applies also in situations involving both regular additive spans and modules.

4.3. Let $\mathcal{A}$ be an additive category and let M be an $\mathcal{A}$-module. It is clear that the system of homomorphisms $\circ$: $\mathrm{Hom}(A', A) \otimes M(A') \to M(A)$ gives

$$\mathrm{Hom}(x, a) \otimes_{\mathcal{A}} M(x) = M(a). \tag{4.3.1}$$

In rewriting the above homomorphisms as $M(A') \to \mathrm{Hom}_Z(\mathrm{Hom}(A', A), M(A))$, we easily get

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}(a, x), M(x)) = M(a). \tag{4.3.2}$$

Dually for an $\mathcal{A}^*$-module M we have

$$M(x) \otimes_{\mathcal{A}} \mathrm{Hom}(a, x) = M(a), \tag{4.3.1*}$$

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}(x, a), M(x)) = M(a). \tag{4.3.2*}$$

Let now $(\mathcal{A}, \mathcal{S})$ be a regular S-category. In §3.5 we have verified the identity $\overline{\mathcal{S}E^q} \ast \overline{\mathcal{S}E^p}(a, c) = \overline{\mathcal{S}E^{p+q}}(a, c)$. In view of Composition Product Theorem II, this now leads to

$$\mathrm{Ext}^q{}_{(\mathcal{A}, \mathcal{S})}(x, c) \otimes_{\mathcal{A}} \mathrm{Ext}^p{}_{(\mathcal{A}, \mathcal{S})}(a, x) = \mathrm{Ext}^{p+q}{}_{(\mathcal{A}, \mathcal{S})}(a, c), \tag{4.3.3}$$

where $[Y] \otimes_{\mathcal{A}} [X]$ corresponds to $[Y] \circ [X] = [Y \bigcirc X]$. Then because of the associativity (4.1.2) we get

$$\mathrm{Ext}^n{}_{(\mathcal{A}, \mathcal{S})} = \mathrm{Ext}^1{}_{(\mathcal{A}, \mathcal{S})} \otimes_{\mathcal{A}} \mathrm{Ext}^1{}_{(\mathcal{A}, \mathcal{S})} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \mathrm{Ext}^1{}_{(\mathcal{A}, \mathcal{S})} \quad (n \text{ times}). \tag{4.3.4}$$

Thus the 'graded' $\mathcal{A}^*$-$\mathcal{A}$-module $\mathrm{Ext}_{(\mathcal{A}, \mathcal{S})} = \mathrm{Hom} \oplus \sum_{n=1}^{\infty} \mathrm{Ext}^n{}_{(\mathcal{A}, \mathcal{S})}$ may be called the 'tensor algebra' of $\mathrm{Ext}^1{}_{(\mathcal{A}, \mathcal{S})}$. In the sequel we shall understand that $\mathrm{Ext}^0{}_{(\mathcal{A}, \mathcal{S})}$ means Hom.

DEFINITION. *The n-th right $\mathcal{S}$-satellite of an $\mathcal{A}$-module M is the $\mathcal{A}$-module*

$$\mathcal{S}^n M(a) = \mathrm{Ext}^n{}_{(\mathcal{A}, \mathcal{S})}(x, a) \otimes_{\mathcal{A}} M(x).$$

The n-th right $\mathcal{S}$-satellite of an $\mathcal{A}^*$-module M is the $\mathcal{A}^*$-module

$$\mathcal{S}^n M(a) = M(x) \otimes_{\mathcal{A}} \mathrm{Ext}^n{}_{(\mathcal{A}, \mathcal{S})}(a, x).$$

The n-th left $\mathcal{S}$-satellite of an $\mathcal{A}$-module M is the $\mathcal{A}$-module

$$\mathcal{S}_n M(a) = \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ext}^n{}_{(\mathcal{A}, \mathcal{S})}(a, x), M(x)).$$

The n-th left $\mathcal{S}$-satellite of an $\mathcal{A}^*$-module M is the $\mathcal{A}^*$-module

$$\mathcal{S}_n M(a) = \mathrm{Hom}_{\mathcal{A}}(\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(x, a), M(x)).$$

In virtue of (4.1.2~4), (4.3.1~4) the following formulas are readily verified:

$$\mathcal{S}^0 M = \mathcal{S}_0 M = M,\ \mathcal{S}^q \mathcal{S}^p M = \mathcal{S}^{p+q} M,\ \mathcal{S}_q \mathcal{S}_p M = \mathcal{S}_{p+q} M,$$
$$\mathcal{S}^q \mathrm{Ext}^p{}_{(\mathcal{A},\mathcal{S})}(A, a) = \mathrm{Ext}^{p+q}{}_{(\mathcal{A},\mathcal{S})}(A, a),$$
$$\mathcal{S}^p \mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(a, C) = \mathrm{Ext}^{p+q}{}_{(\mathcal{A},\mathcal{S})}(a, C).$$

In view of duality principle we shall suppose in the sequel that M is an $\mathcal{A}$-module. In connection with $\mathcal{S}$-satellites we note the following products arising from the composition product $(p, q \geqq 0)$:

$$\circ:\ \mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(B, C) \otimes \mathcal{S}^p M(B) \to \mathcal{S}^{p+q} M(C),$$
$$v^q \circ (u^p \otimes_{\mathcal{A}} m) = (v^q \circ u^p) \otimes_{\mathcal{A}} m$$
$$(m \in M(A),\ u^p \in \mathrm{Ext}^p{}_{(\mathcal{A},\mathcal{S})}(A, B),\ v^q \in \mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(B, C)).$$
$$\backslash:\ \mathrm{Ext}^p{}_{(\mathcal{A},\mathcal{S})}(A, B) \otimes \mathcal{S}_{p+q} M(A) \to \mathcal{S}_q M(B),$$
$$(u^p \backslash \sigma_{p+q}) v^q = \sigma_{p+q}(v^q \circ u^p)$$
$$(\sigma_{p+q} \in \mathcal{S}_{p+q} M(A),\ u^p \in \mathrm{Ext}^p{}_{(\mathcal{A},\mathcal{S})}(A, B),\ v^q \in \mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(B, C)).$$

These satisfy the associativity formulas:

(4.3.5) $$w^r \circ (v^q \circ m^p) = (w^r \circ v^q) \circ m^p$$
$$(m^p \in \mathcal{S}^p M(B),\ v^q \in \mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(B, C),\ w^r \in \mathrm{Ext}^r{}_{(\mathcal{A},\mathcal{S})}(C, D)).$$

(4.3.6) $$v^q \backslash (u^p \backslash \sigma_{p+q+r}) = (v^q \circ u^p) \backslash \sigma_{p+q+r}$$
$$(\sigma_{p+q+r} \in \mathcal{S}_{p+q+r} M(A),\ u^p \in \mathrm{Ext}^p{}_{(\mathcal{A},\mathcal{S})}(A, B),\ v^q \in \mathrm{Ext}^q{}_{(\mathcal{A},\mathcal{S})}(B, C)).$$

Note that for $\varphi \in \mathrm{Ext}^0{}_{(\mathcal{A},\mathcal{S})}(A, B) = \mathrm{Hom}(A, B)$ we have $\varphi \backslash \sigma = \varphi \circ \sigma$.

4.4. Satellites of functors [4] are originally defined using projectives or injectives. We shall now show that our definition of satellites is a suitable generalization. This will mean in particular that our definition of Ext^n is also suitable. We persist in speaking of regular S-categories.

Let $(\mathcal{A}, \mathcal{S})$ be a regular S-category. An object P in $\mathcal{A}$ is called *$\mathcal{S}$-projective* if every special projection $\to P$ is direct. Dually an object Q in $\mathcal{A}$ is called *$\mathcal{S}$-injective* if every special injection $Q \to$ is direct. Our assertion consists in the following:

SATELLITE THEOREM I. *Let n be a positive integer.*

(i_n) *Let*

$$\boldsymbol{P}:\ \emptyset \to A_n \xrightarrow{\alpha_n} P_n \to P_{n-1} \to \cdots \to P_1 \to A \to \emptyset$$

be a special exact sequence with P's $\mathcal{S}$-projective. Then the sequence of additive groups

$$0 \to \mathcal{S}_n M(A) \xrightarrow{[P]\backslash} M(A_n) \xrightarrow{\alpha_n{}^\circ} M(P_n)$$

is exact for every $\mathcal{A}$-module M.

(ii$_n$) *Let*

$$\boldsymbol{Q}: \quad \emptyset \to B \to Q^1 \to Q^2 \to \cdots \to Q^n \xrightarrow{\beta_n} B^n \to \emptyset$$

be a special exact sequence with Q's $\mathcal{S}$-injective. Then the sequence of additive groups

$$M(Q^n) \xrightarrow{\beta_n} M(B^n) \xrightarrow{[\boldsymbol{Q}]^\circ} \mathcal{S}^n M(B) \to 0$$

is exact for every $\mathcal{A}$-module M.

The proof will be completed later in § 4.6. Here we remark the implications (i$_1$), (i$_n$)$\rightharpoonup$(i$_{n+1}$) ; (ii$_1$), (ii$_n$)$\rightharpoonup$(ii$_{n+1}$). Suppose (i$_1$) holds. We note first that $\mathcal{S}_1 M(P)=0$ if P is $\mathcal{S}$-projective. In fact applying (i$_1$) to M and the trivial sequence $\emptyset \to \emptyset \to P \Rightarrow P \to \emptyset$, we obtain the exact sequence $0 \to \mathcal{S}_1 M(P) \to M(\emptyset)$. Since M is additive, $M(\emptyset)=0$, and so $\mathcal{S}_1 M(P)=0$. Next let

$$\boldsymbol{P}: \quad \emptyset \to A_{n+1} \xrightarrow{\alpha_{n+1}} P_{n+1} \to P_n \to \cdots \to P_1 \to A \to \emptyset$$

be a special exact sequence with P's $\mathcal{S}$-projective. This can be decomposed as $\boldsymbol{P}=\boldsymbol{P}^1 \bigcirc \boldsymbol{P}'$, where

$$\boldsymbol{P}': \quad \emptyset \to A_n \to P_n \to P_{n-1} \to \cdots \to P_1 \to A \to \emptyset,$$

$$\boldsymbol{P}^1: \quad \emptyset \to A_{n+1} \xrightarrow{\alpha_{n+1}} P_{n+1} \to A_n \to \emptyset.$$

We now suppose (i$_n$) holds, and apply it to $\mathcal{S}_1 M$ and $\boldsymbol{P}'$. We also apply (i$_1$) to M and $\boldsymbol{P}^1$. We obtain exact sequences

$$0 \to \mathcal{S}_{n+1} M(A) \xrightarrow{[\boldsymbol{P}']\backslash} \mathcal{S}_1 M(A_n) \to \mathcal{S}_1 M(P_n),$$

$$0 \to \mathcal{S}_1 M(A_n) \xrightarrow{[\boldsymbol{P}^1]\backslash} M(A_{n+1}) \xrightarrow{\alpha_{n+1}{}^\circ} M(P_{n+1}).$$

Since $\mathcal{S}_1 M(P_n)=0$, $[\boldsymbol{P}']\backslash$ is an isomorphism. On the other hand by (4.3.6) we have $[\boldsymbol{P}^1]\backslash([\boldsymbol{P}']\backslash)=([\boldsymbol{P}^1]\circ[\boldsymbol{P}'])\backslash=[\boldsymbol{P}]\backslash$, and so the sequence

$$0 \to \mathcal{S}_{n+1} M(A) \xrightarrow{[\boldsymbol{P}]\backslash} M(A_{n+1}) \xrightarrow{\alpha_{n+1}{}^\circ} M(P_{n+1})$$

is exact. Similarly the implication (ii$_1$), (ii$_n$)$\rightharpoonup$(ii$_{n+1}$) is verified using (4.3.5) Thus. the theorem is reduced to (i$_1$) and (ii$_1$).

Remark. In the above reduction we have not referred to $\mathcal{S}$-projectivity of the P's. Thus in (i$_n$) the $\mathcal{S}$-projectives can be replaced by any other class of objects with which (i$_1$) remains valid for every $\mathcal{A}$-module M. As for (ii$_n$), the validity of (ii$_n$) for every M requires Q to be $\mathcal{S}$-injective. The lack of full duality between right $\mathcal{S}$-satellites and left $\mathcal{S}$-satellites is due to the fact that we have fixed the range category $\mathcal{M}$ which is not self-dual. As will be seen later, the

proof of (ii$_1$) is more involved than that of (i$_1$), whereas we have certain theorems on right $\mathcal{S}$-satellites, on which anologues for left $\mathcal{S}$-satellites are not established.

4.5. In the regular additive span $\mathcal{S}E^1$: $\mathcal{S}\mathcal{E}^1\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ a map $X_1\to X_2$ giving $X_1\rightleftarrows X_2$ is, by Lemma 3×2.4, necessarily an equivalence map. Therefore the relation $\rightleftarrows$ being thus symmetric, we have $X_1\sim X_2$ if and only if there exists an equivalence map giving $X_1\rightleftarrows X_2$. The zero element in $\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A, B)$ is represented by the direct exact sequence $O^1(A, B): \emptyset\to B\to A\oplus B\to A\to\emptyset$, and so a special exact sequence $X: \emptyset\to B\to E\to A\to\emptyset$ represents 0 if and only if X is direct. Let $\alpha: A'\to A$ be a map, and suppose $[X]\circ\alpha=0$. Then in the diagram $(X\lrcorner\alpha)$

$$\begin{array}{rccccccccc} X\bigcirc\alpha: & \emptyset & \to & B & \to & E' & \to & A' & \to & \emptyset \\ & & & \Downarrow & & \downarrow & & \downarrow & & \\ X: & \emptyset & \to & B & \to & E & \to & A & \to & \emptyset \end{array}$$

$X\bigcirc\alpha$ is direct, and so there exists a map $A'\to E'$ such that $A'E'A'=A'eA'$. Thus the map $A'\to E$ defined by $A'E=A'E'E$ satisfies $A'EA=A'A$. Conversely suppose in the diagram $(X\lrcorner\alpha)$ there exists a map $A'\to E$ with $A'EA=A'A$. Then $X\bigcirc\alpha$ is direct by Lemma 3×2.2. So we get:

PROPOSITION 4.4. *For a special exact sequence $X: \emptyset\to B\to E\to A\to\emptyset$ and a map $\alpha: A'\to A$ we have $[X]\circ\alpha=0$ if and only if there exists a map $A'\to E$ such shat $A'EA=A'A$.*

COROLLARY. *For any special exact sequence $A: \emptyset\to\dot{A}\xrightarrow{\dot{\alpha}}A\xrightarrow{\bar{\alpha}}\bar{A}\to\emptyset$ we have $[A]\circ\bar{\alpha}=0$, $\dot{\alpha}\circ[A]=0$.*

We say an $\mathcal{A}$-module M is *half $\mathcal{S}$-exact* if for every special exact sequence $A: \emptyset\to\dot{A}\to A\to\bar{A}\to\emptyset$ the sequence of additive groups $M(\dot{A})\to M(A)\to M(\bar{A})$ is exact. Proposition 4.4 is generalized in the following:

PROPOSITION 4.5. *Let M be a half $\mathcal{S}$-exact $\mathcal{A}$-module, and let X; $\emptyset\to B\to E\xrightarrow{\varphi}A\to\emptyset$ be a special exact sequence. For an element $m\in M(A)$ we have $[X]\circ m=0$ (in $\mathcal{S}^1M(B)=\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(x, B)\otimes_{\mathcal{A}}M(x)$) if and only if the pair (X, m) satisfies the condition:*

(L) *There exists an element $l\in M(E)$ such that $\varphi\circ l=m$.*

PROOF. If (X, m) satisfies (L), then we have $[X]\circ m=[X]\circ(\varphi\circ l)=([X]\circ\varphi)\circ l=0$. Now (X, m) satisfies (L) if X is direct. So it will suffice to prove that the property (L) is invariant under similarity $(X, m)\sim(X', m')$, i.e., to show that for the commutative diagram

$$\begin{array}{rccccccccc} X': & \emptyset & \to & B & \to & E' & \xrightarrow{\varphi'} & A' & \to & \emptyset \\ & & & \Downarrow & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\alpha} & & \\ X: & \emptyset & \to & B & \to & E & \xrightarrow{\varphi} & A & \to & \emptyset \end{array}$$

and elements $m \in M(A)$, $m' \in M(A')$ such that $\alpha \circ m' = m$, the pair (X, m) satisfies (L) if and only if (X', m') satisfies (L). In one direction this is obvious. In fact if there exists $l' \in M(E')$ with $\varphi' \circ l' = m'$, then $l = \eta \circ l'$ gives $\varphi \circ l = m$. We now suppose conversely that $l \in M(E)$ gives $\varphi \circ l = m$. Take the direct sum sequence

$$\tilde{X}:\ \emptyset \to B \to A' \oplus E \xrightarrow{\tilde{\varphi}} A' \oplus A \to \emptyset$$

of $O^1(A', \emptyset)$ and X. Define $\tilde{\eta}: E' \to A' \oplus E$ to be the sum of φ' and η. Then we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \boldsymbol{E}: & & \boldsymbol{A}: & & \\
 & & & & \emptyset & & \emptyset & & \\
 & & & & \downarrow & & \downarrow & & \\
X': & \emptyset & \to & B & \longrightarrow & E' & \xrightarrow{\varphi'} & A' & \longrightarrow \emptyset \\
 & & & \| & & \downarrow \tilde{\eta} & & \downarrow c^{\alpha} & \\
X: & \emptyset & \to & B & \to & A' \oplus E & \xrightarrow{\tilde{\varphi}} & A' \oplus A & \to \emptyset, \\
 & & & & & \downarrow & & \downarrow \mathfrak{d}^{\alpha} & \\
 & & & & & A & \Longrightarrow & A & \\
 & & & & & \downarrow & & \downarrow & \\
 & & & & & \emptyset & & \emptyset &
\end{array}$$

where $\boldsymbol{E} = (\varphi \vee \alpha)$ is special exact by Lemma 3×2.9*. We denote by $\tilde{l} \in M(A' \oplus E) = M(A') \oplus M(E)$ the sum $m' \oplus l$, and by $\tilde{m} \in M(A' \oplus A) = M(A') \oplus M(A)$ the sum $m' \oplus m$. Then we have $\tilde{\varphi} \circ \tilde{l} = \tilde{m}$, $c^{\alpha} \circ m' = \tilde{m}$. Because of $(\mathfrak{d}^{\alpha} \circ \tilde{\varphi}) \circ \tilde{l} = \mathfrak{d}^{\alpha} \circ \tilde{m} = \mathfrak{d}^{\alpha} \circ c^{\alpha} \circ m' = 0$, there exists an element $l' \in M(E')$ such that $\tilde{\eta} \circ l' = \tilde{l}$, for M is half exact on $\boldsymbol{E}$. Since c^{α}: $A' \to A' \oplus A$ is a direct injection, so is also $c^{\alpha} \circ = M(c^{\alpha})$. Consequently from $c^{\alpha} \circ (\varphi' \circ l')$ $\tilde{\varphi} \circ \tilde{\eta} \circ l' = \tilde{\varphi} \circ \tilde{l} = \tilde{m} = c^{\alpha} \circ m'$ follows $\varphi' \circ l' = m'$. This shows that (X', m') satisfies (L), and the proposition is proved.

PROPOSITION 4.6. *Let M be a half $\mathcal{S}$-exact $\mathcal{A}$-module, and let*

$$\boldsymbol{B}:\ \emptyset \to \dot{B} \xrightarrow{\dot{\beta}} B \xrightarrow{\bar{\beta}} \bar{B} \to \emptyset$$

be a special exact sequence. Then the sequence

$$M(\dot{B}) \xrightarrow{\dot{\beta}\circ} M(B) \xrightarrow{\bar{\beta}\circ} M(\bar{B}) \xrightarrow{[\boldsymbol{B}]\circ} \mathcal{S}^1 M(\dot{B}) \xrightarrow{\dot{\beta}\circ} \mathcal{S}^1 M(B)$$

is exact.

PROOF. The sequence is exact at $M(B)$, because M is half $\mathcal{S}$-exact. Exactness at $M(\bar{B})$, i.e., $\mathrm{Ker}[\boldsymbol{B}]\circ = \mathrm{Im}\,\bar{\beta}\circ$ is a mere restatement of Proposition 4.5. Because of $\dot{\beta}\circ[\boldsymbol{B}] = 0$, we have $\mathrm{Im}[\boldsymbol{B}]\circ \subset \mathrm{Ker}\,\dot{\beta}\circ$. Now every element of $\mathcal{S}^1 M(\dot{B})$ is of the form $[\dot{X}]\circ m$, where $\dot{X}$ is a special exact sequence $\emptyset \to \dot{B} \xrightarrow{\dot{\varphi}} \dot{E} \to A \to \emptyset$ and $m \in M(A)$. Imbed $\dot{X}$ and $\dot{\beta}$ in the commutative diagram $(\dot{\beta} \ulcorner \dot{X})$

$$\begin{array}{ccccccccc}
\dot{X}: & \emptyset & \to & \dot{B} & \xrightarrow{\dot{\varphi}} & \dot{E} & \to & A & \to \emptyset \\
 & & & \downarrow \dot{\beta} & & \downarrow & & \Downarrow & \\
X: & \emptyset & \to & B & \to & E & \xrightarrow{\varphi} & A & \to \emptyset.
\end{array}$$

By Lemma 3×2.9 this gives rise to the special exact sequence

$$\dot\beta\wedge\dot\psi:\ \emptyset\to\dot B\to B\oplus\dot E\to E\to\emptyset$$

and the commutative diagram

$$\begin{array}{rccccccccc}
\dot X: & \emptyset & \to & \dot B & \longrightarrow & \dot E & \longrightarrow & A & \to & \emptyset \\
 & & & \Uparrow & & \uparrow{\scriptstyle \text{ɔ}} & & \uparrow{\scriptstyle \varphi} & & \\
\dot\beta\wedge\dot\psi: & \emptyset & \to & \dot B & \to & B\oplus\dot E & \to & E & \to & \emptyset \\
 & & & \downarrow{\scriptstyle -e} & & \downarrow{\scriptstyle \text{ɔ}} & & \downarrow{\scriptstyle \bar\eta} & & \\
\boldsymbol{B}: & \emptyset & \to & \dot B & \longrightarrow & B & \longrightarrow & \bar B & \to & \emptyset.
\end{array}$$

Suppose $\dot\beta\circ([\dot X]\circ m)=0$. Then $[X]\circ m=0$, and so by Proposition 4.5, there exists an element $l\in M(E)$ such that $\varphi\circ l=m$. So we get $[\dot X]\circ m=[\dot X]\circ\varphi\circ l=[\dot\beta\wedge\dot\psi]\circ l=-[\boldsymbol{B}]\circ\bar\eta\circ l$ which shows $[\dot X]\circ m\in\mathrm{Im}[\boldsymbol{B}]\circ$. This establishes the proposition.

COROLLARY *Let* $\boldsymbol{B}:\ \emptyset\to\dot B\xrightarrow{\dot\beta}B\xrightarrow{\bar\beta}\bar B\to\emptyset$ *be a special exact sequence. Then for every object* A *the sequences*

$$0\longrightarrow\mathrm{Hom}(A,\dot B)\xrightarrow{\dot\beta\circ}\mathrm{Hom}(A,B)\xrightarrow{\bar\beta\circ}\mathrm{Hom}(A,B)\xrightarrow{[\boldsymbol{B}]\circ}\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A,\dot B)\xrightarrow{\dot\beta\circ}\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A,B),$$

$$0\longrightarrow\mathrm{Hom}(\bar B,A)\xrightarrow{\circ\bar\beta}\mathrm{Hom}(B,A)\xrightarrow{\circ\dot\beta}\mathrm{Hom}(\dot B,A)\xrightarrow{\circ[\boldsymbol{B}]}\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(\bar B,A)\xrightarrow{\circ\bar\beta}\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(B,A)$$

are exact.

4.6. Let P be an $\mathcal{S}$-projective object. Then every special exact sequence X: $\emptyset\to B\to E\to P\to\emptyset$ is direct, and so $\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(P,B)=0$ for every object B. If $\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(P,B)=0$ for every object B, then by the corollary to Proposition 4.6 the covariant functor $\mathrm{Hom}(P,b)$ is exact on special exact sequences, and in particular P must be $\mathcal{S}$-projective. Let $\boldsymbol{P}:\ \emptyset\to A_1\xrightarrow{\alpha_1}P\to A\to\emptyset$ be a special exact sequence with P $\mathcal{S}$-projective. Then by the same corollary we have for every object B the exact sequence

$$\mathrm{Hom}(P,B)\xrightarrow{\circ\alpha_1}\mathrm{Hom}(A_1,B)\xrightarrow{\circ[\boldsymbol{P}]}\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A,B)\to 0 \tag{4.6.1}$$

for we now have $\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(P,B)=0$. Let M be any $\mathcal{A}$-module. We shall prove the assertion (i_1) that the sequence

$$0\longrightarrow\mathcal{S}_1M(A)\xrightarrow{[\boldsymbol{P}]\backslash}M(A_1)\xrightarrow{\alpha_1\circ}M(P)$$

is exact. Let σ be an element of $\mathcal{S}_1M(A)$, i.e. a natural transformation σ: $\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A,x)\to M(x)$. We have $\alpha_1\circ([\boldsymbol{P}]\backslash\sigma)=\alpha_1\circ\sigma(A_1)[\boldsymbol{P}]=\sigma(P)(\alpha_1\circ[\boldsymbol{P}])=0$, and so $\mathrm{Im}[\boldsymbol{P}]\backslash\subset\mathrm{Ker}\,\alpha_1\circ$. Now be (4.6.1) any element $[X]\in\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A,B)$ can be written as $[X]=\varphi_1\circ[\boldsymbol{P}]$ for some $\varphi_1\in\mathrm{Hom}(A_1,B)$. Thus if $[\boldsymbol{P}]\backslash\sigma=\sigma(A_1)[\boldsymbol{P}]=0$, then we get $[X]\backslash\sigma=\sigma(B)[X]=\sigma(B)(\varphi_1\circ[\boldsymbol{P}])=\varphi_1\circ(\sigma(A_1)[\boldsymbol{P}])=0$, and so $\sigma=0$. This proves that $[\boldsymbol{P}]\backslash$ is a monomorphism. Finally let $m_1\in M(A_1)$ be such that $\alpha_1\circ m_1$

$=0$. For each element $[X]\in\mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(A,B)$ we take $\varphi_1\in\mathrm{Hom}(A_1,B)$ such that $[X]=\varphi_1\circ[\boldsymbol{P}]$. Then $\varphi_1\circ m_1\in M(B)$ does not depend on the choice of φ_1, because if $\varphi_1\circ[\boldsymbol{P}]=\varphi_1'\circ[\boldsymbol{P}]$, there exists $\varphi\in\mathrm{Hom}(P,B)$ such that $\varphi_1'-\varphi_1=\varphi\circ\alpha_1$, and so $\varphi_1'\circ m_1-\varphi_1\circ m_1=\varphi\circ\alpha_1\circ m_1=0$. We put $\sigma(B)[X]=\varphi_1\circ m_1$. Then for any map $\beta\colon B\to B'$ we have $\beta\circ(\sigma(B)[X])=\beta\circ\varphi_1\circ m_1$, $\beta\circ[X]=(\beta\circ\varphi_1)\circ[\boldsymbol{P}]$, and so $\beta\circ(\sigma(B)[X])=\sigma(B')(\beta\circ[X])$. Therefore by Proposition 4.1 we get $\sigma\in\mathrm{Hom}_{\mathcal{A}}(\mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(A,x),M(x))=\mathcal{S}_1M(A)$. Since $[\boldsymbol{P}]\backslash\sigma=\sigma(A_1)[\boldsymbol{P}]=\sigma(A_1)(e_{A_1}\circ[\boldsymbol{P}])=e_{A_1}\circ m_1=m_1$, we get $m_1\in\mathrm{Im}[\boldsymbol{P}]\backslash$, and so $\mathrm{Ker}\,\alpha_1\circ=\mathrm{Im}[\boldsymbol{P}]\backslash$. This completes the proof of (i_1), and thus Satellite Theorem I, (i_n) is established.

Let Q be an $\mathcal{S}$-injective object. Then $\mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(A,Q)=0$ for every object A, and so a special exact sequence $\boldsymbol{Q}:\emptyset\to B\to Q\xrightarrow{\beta^1}B^1\to\emptyset$ gives rise to the exact sequence

$$\mathrm{Hom}(A,Q)\xrightarrow{\beta^1\circ}\mathrm{Hom}(A,B^1)\xrightarrow{[\boldsymbol{Q}]\circ}\mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(A,B)\longrightarrow 0. \tag{4.6.2}$$

We shall prove the assertion (ii_1) that the sequence

$$M(Q)\xrightarrow{\beta^1\circ}M(B^1)\xrightarrow{[\boldsymbol{Q}]\circ}\mathcal{S}^1M(B)\longrightarrow 0$$

is exact. Firstly $\mathrm{Im}\,\beta^1\circ\subset\mathrm{Ker}[\boldsymbol{Q}]\circ$ because $[\boldsymbol{Q}]\circ\beta^1=0$. Secondary $[\boldsymbol{Q}]\circ$ is an epimorphism. In fact every element in $\mathcal{S}^1M(B)=\mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(x,B)\otimes_{\mathcal{A}}M(x)$ is of the form $[X]\otimes_{\mathcal{A}}m=[X]\circ m$ ($[X]\in\mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(A,B)$, $m\in M(A)$) for some object A. In virtue of (4.6.2) we have $[X]=[\boldsymbol{Q}]\circ\varphi^1$ for some $\varphi^1\in\mathrm{Hom}(A,B^1)$. Therefore we get $[X]\circ m=[\boldsymbol{Q}]\circ(\varphi^1\circ m)$, and so $[\boldsymbol{Q}]\circ\colon M(B^1)\to\mathcal{S}^1M(B)$ is an epimorphism. Finally let $m^1\in M(B^1)$ be such that $[\boldsymbol{Q}]\circ m^1=0$. This means that for some $k\geqq 0$ there exist a commutative diagram.

$$\begin{array}{rccccccccc}
X_0: & \emptyset & \longrightarrow & B & \longrightarrow & E_0 & \longrightarrow & A_0 & \longrightarrow & \emptyset \\
 & & & \Downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_0} & & \\
 & & & \vdots & & \vdots & & \vdots & & \\
 & & & \Uparrow & & \uparrow & & \uparrow & & \\
X_{2k-2}: & \emptyset & \longrightarrow & B & \longrightarrow & E_{2k-2} & \longrightarrow & A_{2k-2} & \longrightarrow & \emptyset \\
 & & & \Downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_{2k-2}} & & \\
X_{2k-1}: & \emptyset & \longrightarrow & B & \longrightarrow & E_{2k-1} & \longrightarrow & A_{2k-1} & \longrightarrow & \emptyset \\
 & & & \Uparrow & & \uparrow & & \uparrow{\scriptstyle \alpha_{2k-1}} & & \\
X_{2k}: & \emptyset & \longrightarrow & B & \longrightarrow & E_{2k} & \longrightarrow & A_{2k} & \longrightarrow & \emptyset \\
 & & & \Downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_{2k}} & & \\
\boldsymbol{Q}: & \emptyset & \longrightarrow & B & \longrightarrow & Q & \xrightarrow{\beta^1} & B^1 & \longrightarrow & \emptyset
\end{array}$$

with special exact rows, and elements $m_\nu\in M(A_\nu)$ ($\nu=0,1,\cdots,2k$) such that

$$0=m_0,\ \alpha_0\circ m_0=m_1=\alpha_1\circ m_2,\cdots,\ \alpha_{2k-2}\circ m_{2k-2}=m_{2k-1}=\alpha_{2k-1}\circ m_{2k},\ \alpha_{2k}\circ m_{2k}=m^1.$$

We shall prove $m^1\equiv 0\ (\mathrm{mod}\,\mathrm{Im}\,\beta^1\circ)$ by induction on k. If $k=0$ then we have m^1

$=\alpha_0\circ m_0=0$. Suppose $k\geqq 1$. Since Q is $\mathcal{S}$-injective, the functor $\mathrm{Hom}(a, Q)$ is exact on special exact sequences. Therefore there exists a map $E_{2k-1}\to Q$ such that $BE_{2k-1}Q=BQ$, and so we get a commutative diagram

$$\begin{array}{rccccccccc} X_{2k-1}: & \emptyset & \to & B & \to & E_{2k-1} & \to & A_{2k-1} & \to & \emptyset \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\ \boldsymbol{Q}: & \emptyset & \to & B & \to & Q & \to & B^1 & \to & \emptyset. \end{array}$$

Since we have $BE_{2k}\cdot(E_{2k}Q-E_{2k}E_{2k-1}Q)=BQ-BQ=0$, there is a map $\varphi: A_{2k}\to Q$ such that $E_{2k}A_{2k}Q=E_{2k}Q-E_{2k}E_{2k-1}Q$. Then from $E_{2k}A_{2k}QB^1=E_{2k}QB^1-E_{2k}E_{2k-1}QB^1$ $=E_{2k}A_{2k}B^1-E_{2k}E_{2k-1}A_{2k-1}B^1=E_{2k}A_{2k}\cdot(A_{2k}B^1-A_{2k}A_{2k-1}B^1)$ follows $A_{2k}QB^1=A_{2k}B^1$ $-A_{2k}A_{2k-1}B^1$, i.e. $\beta^1\circ\varphi=\alpha_{2k}-\alpha\circ\alpha_{2k-1}$. Consequently we have $\beta^1\circ\varphi\circ m_{2k}=\alpha_{2k}\circ m_{2k}$ $-\alpha\circ\alpha_{2k-1}m_{2k}=m^1-\alpha\circ\alpha_{2k-2}\circ m_{2k-2}$, and so $m^1\equiv\alpha\circ\alpha_{2k-2}\circ m_{2k-2}(\mathrm{mod}\,\mathrm{Im}\beta^1\circ)$. Since the composition of $X_{2k-2}\to X_{2k-1}\to \boldsymbol{Q}$ gives the commutative diagram

$$\begin{array}{rccccccccc} X_{2k-2}: & \emptyset & \to & B & \to & E_{2k-2} & \to & A_{2k-2} & \to & \emptyset \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle\alpha\circ\alpha_{2k-2}} & & \\ \boldsymbol{Q}: & \emptyset & \to & B & \to & Q & \xrightarrow{\beta^1} & B^1 & \longrightarrow & \emptyset, \end{array}$$

we get $\alpha\circ\alpha_{2k-2}\circ m_{2k-2}\equiv 0$ $(\mathrm{mod}\,\mathrm{Im}\,\beta^1\circ)$ by induction hyphothesis. Hence we obtain $m^1\equiv 0$ $(\mathrm{mod}\,\mathrm{Im}\,\beta^1\circ)$, and so $\mathrm{Ker}[\boldsymbol{Q}]\circ=\mathrm{Im}\,\beta^1\circ$. Thus (ii_1), and therefore (ii_n), is verified. This completes the proof of Satellite Theorem I.

4.7. Let $A: \emptyset\to\dot{A}\xrightarrow{\dot{\alpha}}A\xrightarrow{\bar{\alpha}}\bar{A}\to\emptyset$ be a special exact sequence, and let M be any $\mathcal{A}$-module. By successive application of $\dot{\alpha}\circ$, $\bar{\alpha}\circ$, $[A]\backslash$ we get the unbounded sequence of left $\mathcal{S}$-satellites

$$\begin{array}{rccccc} \mathcal{S}_\infty M(A): & & \cdots & & \xrightarrow{\bar{\alpha}\circ} & \mathcal{S}_2M(\bar{A}) \\ & \xrightarrow{[A]\backslash} & \mathcal{S}_1M(\dot{A}) & \xrightarrow{\dot{\alpha}\circ} \mathcal{S}_1M(A) & \xrightarrow{\bar{\alpha}\circ} & \mathcal{S}_1M(\bar{A}) \\ & \xrightarrow{[A]\backslash} & M(\dot{A}) & \xrightarrow{\dot{\alpha}\circ} M(A) & \xrightarrow{\bar{\alpha}\circ} & M(\bar{A}). \end{array}$$

This sequence $\mathcal{S}_\infty M(A)$ is of order 2, for we have $\bar{\alpha}\circ\dot{\alpha}=0$, $\dot{\alpha}\circ([A]\backslash\sigma)=\dot{\alpha}\backslash([A]\backslash\sigma)$ $=(\dot{\alpha}\circ[A])\backslash\sigma=0$, and $[A]\backslash(\bar{\alpha}\circ\sigma)=[A]\backslash(\bar{\alpha}\backslash\sigma)=([A]\circ\bar{\alpha})\backslash\sigma=0$. Similarly by successive application of $\dot{\alpha}\circ$, $\bar{\alpha}\circ$, $[A]\circ$ we get the unbounded sequence of right $\mathcal{S}$-satellites

$$\begin{array}{rccccc} \mathcal{S}^\infty M(A): & & M(\dot{A}) & \xrightarrow{\dot{\alpha}\circ} M(A) & \xrightarrow{\bar{\alpha}\circ} & M(\bar{A}) \\ & \xrightarrow{[A]\circ} & \mathcal{S}^1M(\dot{A}) & \xrightarrow{\dot{\alpha}\circ} \mathcal{S}^1M(A) & \xrightarrow{\bar{\alpha}\circ} & \mathcal{S}^1M(\bar{A}) \\ & \xrightarrow{[A]\circ} & \mathcal{S}^2M(\dot{A}) & \xrightarrow{\dot{\alpha}\circ} \cdots. & & \end{array}$$

Again the sequence $\mathcal{S}^\infty M(A)$ is of order 2, for we have $\bar{\alpha}\circ\dot{\alpha}=0$, $[A]\circ\bar{\alpha}=0$, and $\dot{\alpha}\circ[A]=0$. In view of [4, III-3] it would be desirable to have exactness of $\mathcal{S}_\infty M(A)$

and $\mathcal{S}^\infty M(A)$ whenever M is half $\mathcal{S}$-exact. On this point we have we have the following theorems:

SATELLITE THEOREM II. *For a half $\mathcal{S}$-exact $\mathcal{A}$-module M the right satellite sequence $\mathcal{S}^\infty M(A)$ is exact if $(\mathcal{A}, \mathcal{S})$ is quasi-abelian.*

SATELLITE THEOREM III. *For a half $\mathcal{S}$-exact $\mathcal{A}$-module M the left satellite sequence $\mathcal{S}_\infty M(A)$ is exact if $(\mathcal{A}, \mathcal{S})$ is quasi-abelian and if every object in $\mathcal{A}$ admits a special projection from an $\mathcal{S}$-projective object.*

As a corollary to Satellite Theorem II and its dual we get the result that the sequences

$$\begin{aligned}\mathrm{Ext}_{(\mathcal{A},\mathcal{S})}(C, A):\ 0 \longrightarrow\ & \mathrm{Hom}(C, \dot{A}) \xrightarrow{\dot{\alpha}\circ} \mathrm{Hom}(C, A) \xrightarrow{\bar{\alpha}\circ} \mathrm{Hom}(C, \bar{A}) \\ & \xrightarrow{[A]^\frown} \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(C, \dot{A}) \xrightarrow{\dot{\alpha}\circ} \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(C, A) \xrightarrow{\bar{\alpha}\circ} \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(C, \bar{A}) \\ & \xrightarrow{[A]^\frown} \mathrm{Ext}^2_{(\mathcal{A},\mathcal{S})}(C, \dot{A}) \xrightarrow{\dot{\alpha}\circ} \cdots,\end{aligned}$$

$$\begin{aligned}\mathrm{Ext}_{(\mathcal{A},\mathcal{S})}(A, B):\ 0 \longrightarrow\ & \mathrm{Hom}(\bar{A}, B) \xrightarrow{\circ\bar{\alpha}} \mathrm{Hom}(A, B) \xrightarrow{\circ\dot{\alpha}} \mathrm{Hom}(\dot{A}, B) \\ & \xrightarrow{\circ[A]} \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(\bar{A}, B) \xrightarrow{\circ\bar{\alpha}} \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(A, B) \xrightarrow{\circ\dot{\alpha}} \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(\dot{A}, B) \\ & \xrightarrow{\circ[A]} \mathrm{Ext}^2_{(\mathcal{A},\mathcal{S})}(\bar{A}, B) \xrightarrow{\circ\bar{\alpha}} \cdots\end{aligned}$$

are exact if $(\mathcal{A}, \mathcal{S})$ is quasi-abelian.

PROOF OF SATELLITE THEOREM II. For convenience we replace A by B: $\emptyset \to \dot{B} \xrightarrow{\dot{\beta}} B \xrightarrow{\bar{\beta}} \bar{B} \to \emptyset$. Because of $\mathcal{S}^1\mathcal{S}^n = \mathcal{S}^{n+1}$ it will suffice to show exactness in the first two rows of $\mathcal{S}^\infty M(B)$. In virtue of Proposition 4.6 we have already the exact sequence $M(\dot{B}) \xrightarrow{\dot{\beta}\circ} M(B) \xrightarrow{\bar{\beta}\circ} M(\bar{B}) \xrightarrow{[B]^\frown} \mathcal{S}^1 M(\dot{B}) \xrightarrow{\dot{\beta}\circ} \mathcal{S}^1 M(B)$. So it remains only to prove $\mathrm{Ker}\,\bar{\beta}\circ \subset \mathrm{Im}\,\dot{\beta}\circ$ in the sequence $\mathcal{S}^1 M(\dot{B}) \xrightarrow{\dot{\beta}\circ} \mathcal{S}^1 M(B) \xrightarrow{\bar{\beta}\circ} \mathcal{S}^1 M(\bar{B})$. Now every element in $\mathcal{S}^1 M(B) = \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(x, B) \otimes_{\mathcal{A}} M(x)$ is of the form $[X]\circ m$ ($[X] \in \mathrm{Ext}^1_{(\mathcal{A},\mathcal{S})}(A, B)$, $m \in M(A)$) for some A. Suppose $\bar{\beta}\circ[X]\circ m = 0$. Then in the commutative diagram ($\bar{\beta}\ulcorner X$)

$$\begin{array}{lccccccccc} X: & \emptyset & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & \emptyset \\ & & & \bar{\beta}\big\downarrow & & \big\downarrow & & \big\Downarrow & & \\ \bar{\beta}\bigcirc X: & \emptyset & \longrightarrow & \bar{B} & \longrightarrow & \bar{E} & \xrightarrow{\bar{\varphi}} & A & \longrightarrow & \emptyset \end{array}$$

we have $[\bar{\beta}\bigcirc X]\circ m = 0$, and so by Proposition 4.5 there exists an element $\bar{l} \in M(\bar{E})$ such that $\bar{\varphi}\circ\bar{l} = m$. By Lemma 3×3.3, (iv)* the sequence E: $\emptyset \to \dot{B} \to E \to \bar{E} \to \emptyset$ ($\dot{B}E = \dot{B}BE$) is special exact. Then from the commutative diagram

$$\begin{array}{lccccccccc} E: & \emptyset & \longrightarrow & \dot{B} & \longrightarrow & E & \longrightarrow & \bar{E} & \longrightarrow & \emptyset \\ & & & \dot{\beta}\big\downarrow & & \big\Downarrow & & \big\downarrow\bar{\varphi} & & \\ X: & \emptyset & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & \emptyset \end{array}$$

we obtain $[X]\circ\bar{\varphi}=\dot{\beta}\circ[\boldsymbol{E}]$, and so $[X]\circ m=[X]\circ\bar{\varphi}\circ\bar{l}=\dot{\beta}\circ[\boldsymbol{E}]\circ\bar{l}$. This proves $[X]\circ m \in \operatorname{Im}\dot{\beta}\circ$, and the proof is completed.

PROOF OF SATELLITE THEOREM III. Because of $\mathcal{S}_1\mathcal{S}_n=\mathcal{S}_{n+1}$ it will suffice to show exactness in the last two rows of $\mathcal{S}_\infty M(A)$. Under our assumption we can take Satellite Theorem I, (i_1) as the definition of $\mathcal{S}_1M$, and the argument in [4, III–3] works. We shall rewrite the proof in terms of our definition
$\mathcal{S}_1M(a)=\operatorname{Hom}_{\mathcal{A}}(\operatorname{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(a, x), M(x))$.

We shall firstly show that if there exists a special exact sequence $\bar{\boldsymbol{P}}\colon \emptyset\longrightarrow\bar{A}_1 \xrightarrow{\bar{\rho}_1}\bar{P}\longrightarrow\bar{A}\longrightarrow\emptyset$ with $\bar{P}$ $\mathcal{S}$-projective, then the sequence

$$\text{(4.7.1)}\qquad \mathcal{S}_1M(\bar{A})\xrightarrow{[A]\backslash}M(\dot{A})\xrightarrow{\dot{\alpha}\circ}M(A)$$

is exact. Since $\bar{P}$ is $\mathcal{S}$-projective there exists a map $\bar{P}\to A$ such that $\bar{P}A\bar{A}=\bar{P}\bar{A}$. So there is also a map $\bar{\alpha}_1\colon \bar{A}_1\to\dot{A}$ such that $\bar{A}_1\dot{A}A=\bar{A}_1\bar{P}A$, and then we have $\bar{\alpha}_1\circ[\boldsymbol{P}]=[\boldsymbol{A}]$. Define a map $\rho\colon \bar{P}\oplus\dot{A}\to A$ by $(\bar{P}\oplus\dot{A})A=(\bar{P}\oplus\dot{A})\text{ɔ}\bar{P}A+(\bar{P}\oplus\dot{A})\text{ɔ}\dot{A}A$, and $\rho_1\colon \bar{A}_1\to\bar{P}\oplus\dot{A}$ by $\bar{A}_1(\bar{P}\oplus\dot{A})=\bar{A}_1\bar{P}c(\bar{P}\oplus\dot{A})-\bar{A}_1\dot{A}c(\bar{P}\oplus\dot{A})$. Then the diagram

$$\begin{tikzcd}
 & & & \bar{A}_1 \arrow[r, Rightarrow] \arrow[d, "\rho_1"] & \bar{A}_1 \arrow[d, "\bar{\rho}_1"] & \\
 & \emptyset \arrow[r] & \dot{A} \arrow[r, "c"] \arrow[d, Rightarrow] & \bar{P}\oplus\dot{A} \arrow[r, "\text{ɔ}"] \arrow[d, "\rho"] & \bar{P} \arrow[r] \arrow[d] & \emptyset \\
\boldsymbol{A}\colon & \emptyset \arrow[r] & \dot{A} \arrow[r] & A \arrow[r] & \bar{A} \arrow[r] & \emptyset
\end{tikzcd}$$

is commutative, and $\bar{A}_1(\bar{P}\oplus\dot{A})A=0$. Therefore by Lemma $3\times 2.10^*$ the sequence $\emptyset\to\bar{A}_1\xrightarrow{\rho_1}\bar{P}\oplus\dot{A}\xrightarrow{\rho}A\to\emptyset$ is special exact. Suppose m is an element in $M(\dot{A})$ such that $\dot{\alpha}\circ m=0$. Then $\rho\circ c\circ m=0$, and so there exists an element $\bar{m}_1\in M(\bar{A}_1)$ such that $\rho_1\circ\bar{m}_1=-c\circ m$. From this we get $\bar{\rho}_1\circ\bar{m}_1=0$ and $\bar{\alpha}_1\circ\bar{m}_1=m$. Therefore by Satellite Theorem I, (i_1) there is an element $\sigma\in\mathcal{S}_1M(\bar{A})$ such that $[\bar{\boldsymbol{P}}]\backslash\sigma=\bar{m}_1$, and so we obtain $m=\bar{\alpha}_1\circ([\bar{\boldsymbol{P}}]\backslash\sigma)=(\bar{\alpha}_1\circ[\bar{\boldsymbol{P}}]\backslash\sigma=[\boldsymbol{A}]\backslash\sigma$. This proves exactness of (4.7.1).

Secondly we shall show that if there exists a special exact sequence $\boldsymbol{P}\colon \emptyset\to A_1 \xrightarrow{\alpha_1}P\to A\to\emptyset$ with P $\mathcal{S}$-projective, and if $(\mathcal{A},\mathcal{S})$ is quasi-abelian, then the sequence

$$\text{(4.7.2)}\qquad \mathcal{S}_1M(A)\xrightarrow{\bar{\alpha}\circ}\mathcal{S}_1M(\bar{A})\xrightarrow{[A]\backslash}M(\dot{A})$$

is exact. Since $PA\bar{A}$ is a special projection we have a special exact sequence $\boldsymbol{P}'$: $\emptyset\to A'\xrightarrow{\rho'}P\to\bar{A}\to\emptyset$ $(P\bar{A}=PA\bar{A})$. This is imbedded in the commutative diagram

$$\begin{tikzcd}
 & & A_1 \arrow[r, Rightarrow] \arrow[d, "\alpha_1'"] & A_1 \arrow[d, "\alpha_1"] & & \\
\boldsymbol{P}'\colon & \emptyset \arrow[r] & A' \arrow[r] \arrow[d, "\alpha'"'] & P \arrow[r] \arrow[d] & \bar{A} \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
\boldsymbol{A}\colon & \emptyset \arrow[r] & \dot{A} \arrow[r, "\dot{\alpha}"] & A \arrow[r, "\bar{\alpha}"] & \bar{A} \arrow[r] & \emptyset ,
\end{tikzcd}$$

and by Lemma 3×3.3, (iii*) the sequence $\emptyset\to A_1\to A'\to\dot{A}\to\emptyset$ is special exact. Suppose $\bar{\sigma}$ is an element in $\mathcal{S}_1M(\bar{A})$ such that $[A]\backslash\bar{\sigma}=0$. Put $m'=[\boldsymbol{P}']\backslash\bar{\sigma}\in M(A')$. Then we have $\alpha'\circ m'=(\alpha'\circ[\boldsymbol{P}'])\backslash\bar{\sigma}=[A]\backslash\bar{\sigma}=0$, and so there exists an element $m_1\in M(A_1)$ such that $\alpha_1'\circ m_1=m'$. On the other hand we have $\rho'\circ m'=(\rho'\circ[\boldsymbol{P}'])\backslash\bar{\sigma}=0$, and so $\alpha_1\circ m_1=\alpha_1'\circ\rho'\circ m'=0$. Therefore by Satellite Theorem I, (i_1) there is an element $\sigma\in\mathcal{S}_1M(A)$ such that $[\boldsymbol{P}]\backslash\sigma=m_1$. Now in virtue of the commutative diagram

$$\begin{array}{rccccccccc}
\boldsymbol{P}: & \emptyset & \longrightarrow & A_1 & \longrightarrow & P & \longrightarrow & A & \longrightarrow & \emptyset \\
 & & & \Big\downarrow{\scriptstyle\alpha_1'} & & \Big\Downarrow & & \Big\downarrow{\scriptstyle\bar{\alpha}} & & \\
\boldsymbol{P}': & \emptyset & \longrightarrow & A' & \longrightarrow & P & \longrightarrow & \bar{A} & \longrightarrow & \emptyset
\end{array}$$

we have $[\boldsymbol{P}']\circ\bar{\alpha}=\alpha_1'\circ[\boldsymbol{P}]$. So we get

$$\begin{aligned}[]
[\boldsymbol{P}']\backslash(\bar{\alpha}\circ\sigma)&=[\boldsymbol{P}']\backslash(\bar{\alpha}\backslash\sigma)=([\boldsymbol{P}']\circ\bar{\alpha})\backslash\sigma=(\alpha_1'\circ[\boldsymbol{P}])\backslash\sigma\\
&=\alpha_1'\circ([\boldsymbol{P}]\backslash\sigma)=\alpha_1'\circ m_1=m'=[\boldsymbol{P}']\backslash\bar{\sigma}.
\end{aligned}$$

Since $[\boldsymbol{P}']\backslash$: $\mathcal{S}_1M(\bar{A})\to M(A')$ is a monomorphism by Satellite Theorem I, (i_1), we obtain $\bar{\alpha}\circ\sigma=\bar{\sigma}$. This proves exactness of (4.7.2).

Finally we put ourselves in the same situation as above. Having proved the exactness of (4.7.1), we may assume that for the special exact sequence A': $\emptyset\to A_1 \xrightarrow{\alpha_1'} A'\to\dot{A}\to\emptyset$ the sequence

(4.7.1′) $$\mathcal{S}_1M(\dot{A})\xrightarrow{[A']\backslash}M(A_1)\xrightarrow{\alpha_1'\circ}M(A')$$

is exact. We now prove the exactness of the sequence

(4.7.3) $$\mathcal{S}_1M(\dot{A})\xrightarrow{\dot{\alpha}\circ}\mathcal{S}_1M(A)\xrightarrow{\bar{\alpha}\circ}\mathcal{S}_1M(\bar{A}).$$

Suppose σ is an element in $\mathcal{S}_1M(A)$ such that $\bar{\alpha}\circ\sigma=0$. Put $m_1=[\boldsymbol{P}]\backslash\sigma\in M(A_1)$. Then we have $\alpha_1'\circ m_1=(\alpha_1'\circ[\boldsymbol{P}])\backslash\sigma=([\boldsymbol{P}']\circ\bar{\alpha})\backslash\sigma=[\boldsymbol{P}']\backslash(\bar{\alpha}\circ\sigma)=0$. Therefore by (4.7.1′) there exists an element $\dot{\sigma}\in\mathcal{S}_1M(\dot{A})$ such that $[A']\backslash\dot{\sigma}=m_1$. Then we get $[\boldsymbol{P}]\backslash(\dot{\alpha}\circ\dot{\sigma})=([\boldsymbol{P}]\circ\dot{\alpha})\backslash\dot{\sigma}=[A']\backslash\dot{\sigma}=m_1=[\boldsymbol{P}]\backslash\sigma$. Since $[\boldsymbol{P}]$ is a monomorphism, we obtain $\dot{\alpha}\circ\dot{\sigma}=\sigma$. Thus (4.7.3) is exact, and the proof is completed.

§5. Similar exact sequences.

5.0. Let $(\mathcal{A},\mathcal{S})$ be a regular S-category. In the regular span $\mathcal{S}E^1$: $\mathcal{S}\mathcal{E}^1\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ we have $X\sim X'$ if and only if there exists an equivalence map $X\to X'$ giving $X\simeq X'$. Let X: $\emptyset\to B\to E\to A\to\emptyset$, X': $\emptyset\to B'\to E'\to A'\to\emptyset$ be special exact sequences, and suppose for maps α: $A\to A'$, β: $B\to B'$ we have $\beta\circ[X]=[X']\circ\alpha$. Then there exists an equivalence map $\beta\bigcirc X\to X'\bigcirc\alpha$ over $(e_A, e_{B'})$, and so by composing this with the natural maps $X\to\beta\bigcirc X$, $X'\bigcirc\alpha\to X'$, we obtain a map $X\to X'$ over (α,β). Thus existence of a map $X\to X'$ over (α,β) is a necessary and sufficient condition

for $\beta\circ[X]=[X']\circ\alpha$.

The situation is not so simple for $\mathcal{S}E^n$: $\mathcal{SE}^n\mathcal{A}\to\mathcal{A}\times\mathcal{A}$ $(n\geqq 2)$. In fact suppose we have a non-direct special exact sequence X: $\emptyset\to B\to E\to A\to\emptyset$. Let

$$X^\natural:\quad \emptyset\to B\to E\oplus B\to A\oplus E\to A\to\emptyset$$

be the direct sum sequence of $X^\sharp$: $\emptyset\to\emptyset\to B\to E\to A\to\emptyset$ and $X^\flat$: $\emptyset\to B\to E\to A\to\emptyset\to\emptyset$. The canonical projection $X^\natural\to X^\flat$ is a map over $(0^A, e_B)$, and so we have $[X^\natural]=[X^\flat]\circ 0^A=0\in\mathrm{Ext}^2{}_{(\mathcal{A},\mathcal{S})}(A, B)$. Thus $X^\natural$ is similar to the trivial sequence $O^2(A, B)$: $\emptyset\to B\Rightarrow B\overset{0}{\to}A\Rightarrow A\to\emptyset$. However we have neither $X^\natural\underset{\to}{\simeq}O^2(A, B)$ nor $O^2(A, B)\underset{\to}{\simeq}X^\natural$, because either one of these relations would imply that X be direct. Here $X^\natural$ and $O^2(A, B)$ are connected by the relations

$$X^\natural\underset{\leftarrow}{\simeq}X_1^\sharp\underset{\to}{\simeq}O^2(A, B),\quad X^\natural\underset{\to}{\simeq}X_1{}^\flat\underset{\leftarrow}{\simeq}O^2(A, B)$$
$$(X_1^\sharp=X^\sharp\oplus O^2(\emptyset, B),\quad X_1{}^\flat=X^\flat\oplus O^2(A, \emptyset)).$$

It is not difficult to see (cf. [10]) that when $\mathcal{A}$ has enough $\mathcal{S}$-projectives ($\mathcal{S}$-injectives) two objects X, X' in $\mathcal{SE}^n\mathcal{A}$ are similar if and only if there exists a third object $\hat{X}$ $(\check{X})$ such that $X\underset{\leftarrow}{\simeq}\hat{X}\underset{\to}{\simeq}X'$ $(X\underset{\to}{\simeq}\check{X}\underset{\leftarrow}{\simeq}X')$. Thence naturally arises the question whether this remains valid without $\mathcal{S}$-projectives ($\mathcal{S}$-injectives). In this section this will be answered affirmatively for quasi-abelian S-categories. Throughout the rest of the paper we assume $(\mathcal{A}, \mathcal{S})$ is a quasi-abelian S-category. M will stand for a half $\mathcal{S}$-exact $\mathcal{A}$-module. Recall that the $\mathcal{A}$-modules $\mathrm{Ext}^n{}_{(\mathcal{A},\mathcal{S})}(A, a)$, $\mathcal{S}^nM(a)$ are half $\mathcal{S}$-exact.

5.1. For convenience' sake we call objects in $\mathcal{SE}^1\mathcal{A}$ *1-blocks*. In writting down a 1-block (i.e. a short special exact sequence) we shall omit the outer $\emptyset$'s as X: $B\to E\to A$. The three objects in X will now be located as $X(+)=B$, $X(0)=E$, $X(-)=A$. Let $X\to X'$ be a map of 1-blocks, and suppose $XX'(t_1)$ $(t_1=+, 0, -)$ are special maps. Using Lemma 3×3.4 and its dual, we see easily that XX' is a proper map (in $\mathcal{SE}^1\mathcal{A}$) if and only if either one (and therefore each) of the three sequences

$$\begin{array}{ll}\mathrm{Ker}\,XX'(*): & \mathrm{Ker}\,XX'(+)\to\mathrm{Ker}\,XX'(0)\to\mathrm{Ker}\,XX'(-),\\ \mathrm{Coker}\,XX'(*): & \mathrm{Coker}\,XX'(+)\to\mathrm{Coker}\,XX'(0)\to\mathrm{Coker}\,XX'(-),\\ \mathrm{Im}\,XX'(*): & \mathrm{Im}\,XX'(+)\to\mathrm{Im}\,XX'(0)\to\mathrm{Im}\,XX'(-)\end{array}$$

is a 1-block. In this case these three 1-blocks make $\mathrm{Ker}\,XX'$, $\mathrm{Coker}\,XX'$, $\mathrm{Im}\,XX'$ respectively. It is readily seen that the totality $\mathcal{S}^{(1)}$ of proper maps $X\to X'$ in $\mathcal{SE}^1\mathcal{A}$ having special constituents $XX'(t_1)$ gives a quasi-abelian S-category $(\mathcal{SE}^1\mathcal{A}, \mathcal{S}^{(1)})$. In this S-category a 1-block means a commutative diagram

$$
\begin{array}{rccccc}
 & \dot{B} & \rightarrow & \dot{E} & \rightarrow & \dot{A} \\
 & \downarrow & & \downarrow & & \downarrow \\
W: & B & \rightarrow & E & \rightarrow & A \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \bar{B} & \rightarrow & \bar{E} & \rightarrow & \bar{A}
\end{array}
$$

comprising six 1-blocks in $(\mathcal{A}, \mathcal{S})$. It will be called a *2-block* in $(\mathcal{A}, \mathcal{S})$. W may be considered as a 1-block of three rows or as a 1-block of three columns. The nine objects in W will be located by a pair $t=(t_1, t_2)$ of symbols $t_i=+, 0, -$. Thereby we take the horizontal direction as the first direction and the vertical direction as the second. By inscribing $*$ we shall designate one of the six 1-blocks in W. Namely $W(*, t_2)$ refers to the horizontal 1-block $W(+, t_2)\rightarrow W(0, t_2)\rightarrow W(-, t_2)$ and $W(t_1, *)$ refers to the vertical 1-block $W(t_1, +)\rightarrow W(t_1, 0)\rightarrow W(t_1, -)$. Also we denote by $W(1, 2)$, $W(2, 1)$ the special exact sequences.

$$W(1, 2)=W(*, +)\bigcirc W(-, *), \qquad W(2, 1)=W(+, *)\bigcirc W(*, -).$$

In $\mathcal{S}\mathcal{E}^n\mathcal{A}$ $(n\geqq 2)$ we shall say that a map $X\rightarrow X'$ is *special* if, in the decomposition of XX' into maps of 1-blocks as $XX'=X_nX_n'\bigcirc\cdots\bigcirc X_1X_1'$, the maps $X_nX_n', \cdots, X_1X_1'$ are all special. Again the totality $\mathcal{S}^{(n)}$ of special maps in $\mathcal{S}\mathcal{E}^n\mathcal{A}$ gives rise to a quasi-abelian S-category $(\mathcal{S}\mathcal{E}^n\mathcal{A}, \mathcal{S}^{(n)})$. A 1-block W in this category means thus a special exact sequence

$$\boldsymbol{W}:\ \emptyset\rightarrow W_+\rightarrow W_n\rightarrow\cdots\rightarrow W_1\rightarrow W_-\rightarrow\emptyset$$

of 1-blocks.

5.2. Let X: $B\xrightarrow{\psi}E\xrightarrow{\varphi}A$, X': $B\xrightarrow{\psi'}E'\xrightarrow{\varphi'}A'$ be 1-blocks. Applying Lemma 3×2.10 to the translation $(\psi'\ulcorner X)$: $X\rightarrow\psi'\bigcirc X$ we obtain a 2-block

$$
\begin{array}{rccccc}
 & B & \longrightarrow & E & \longrightarrow & A \\
 & \downarrow & & \downarrow & & \Vert \\
(X'\wedge X): & E' & \longrightarrow & \hat{A} & \xrightarrow{\hat{\varphi}} & A \\
 & \downarrow & & \downarrow{\scriptstyle \hat{\varphi}'} & & \downarrow \\
 & A' & \Longrightarrow & A' & \longrightarrow & \emptyset\,.
\end{array}
$$

Define a sequence

$$X'\frown X:\ B\longrightarrow\hat{A}\xrightarrow{\tilde{\varphi}}A\oplus A'$$

by $B\hat{A}=BE\hat{A}=BE'\hat{A}$, $\hat{A}(A\oplus A')=\hat{A}Ac_1(A\oplus A')+\hat{A}A'c_2(A\oplus A')$. Then the diagram

$$
\begin{array}{ccccc}
B & \Longrightarrow & B & & \\
\downarrow & & \downarrow & & \\
E' & \longrightarrow & \hat{A} & \longrightarrow & A \\
\downarrow & & \downarrow{\scriptstyle\tilde{\varphi}} & & \Vert \\
A' & \xrightarrow{c_2} & A\oplus A' & \xrightarrow{\partial_1} & A
\end{array}
$$

is commutative, and so by Lemma 3×3.3, (iv*), $X'\frown X$ is a 1-block. Clearly we have

$$[X'\frown X]\circ c_1=[X],\ [X'\frown X]\circ c_2=[X'].$$

We further define

$$X'\sqcap X:\ B\rightarrow E\oplus E'\rightarrow \hat{A}$$

by $B(E\oplus E')=BEc(E\oplus E')-BE'c(E\oplus E')$, $(E\oplus E')\hat{A}=(E\oplus E')\text{ɔ}E\hat{A}+(E\oplus E')\text{ɔ}E'\hat{A}$. By Lemma 3×2.9, $X'\sqcap X$ $(=\psi'\wedge\psi)$ is a 1-block, and because of the commutative diagram

$$\begin{tikzcd}
X: & B \arrow[r] & E \arrow[r] & A \\
X'\sqcap X: & B \arrow[r] \arrow[u, Rightarrow] \arrow[d, "-e"] & E\oplus E' \arrow[r] \arrow[u, "\text{ɔ}"] \arrow[d, "\text{ɔ}"] & A \arrow[u, "\hat{\varphi}"] \arrow[d, "\hat{\varphi}'"] \\
X': & B \arrow[r] & E' \arrow[r] & A'
\end{tikzcd}$$

we have

$$[X]\circ\hat{\varphi}=[X'\sqcap X]=-[X']\circ\hat{\varphi}'.$$

Let

$$\begin{tikzcd}
& B \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & A_1 \arrow[d] \\
W: & E_2' \arrow[r] \arrow[d] & \hat{E} \arrow[r] \arrow[d] & E_1 \arrow[d] \\
& A_1' \arrow[r] & E_1' \arrow[r] & A
\end{tikzcd}$$

be a 2-block. We denote the 1-blocks $W(*,+): B\rightarrow E_2\rightarrow A_1$, $W(+,*): B\rightarrow E_2'\rightarrow A_1'$ by X_2, X_2' respectively. $\hat{A}_1$ will stand for the central term $(X_2'\wedge X_2)(0,0)$ of the 2-block $(X_2'\wedge X_2)$.

LEMMA 3×3.5. *There is a unique map $\hat{A}_1\rightarrow\hat{E}$ with which the identity maps of X_2, X_2' are extended to a map of 2-blocks $(X_2'\wedge X_2)\rightarrow W$. Moreover the sequence*

$$\overline{W}:\ \hat{A}_1\rightarrow\hat{E}\rightarrow A\ \ (\hat{E}A=\hat{E}E_1A=\hat{E}E_1'A)$$

is a 1-block.

PROOF. By Lemma 3×2.6 applied to the upper halves of $(X_2'\wedge X_2)$ and W, there is a unique map $\hat{A}_1\rightarrow\hat{E}$ such that

$$\begin{tikzcd}
E_2' \arrow[r] \arrow[d, Rightarrow] & \hat{A}_1 \arrow[r] \arrow[d] & A_1 \arrow[d] \\
E_2' \arrow[r] & \hat{E} \arrow[r] & E_1
\end{tikzcd}$$

is commutative and $E_2\hat{A}_1\hat{E}=E_2\hat{E}$. Then by Lemma 3×3.3, (iv) the sequence $\overline{W}$ is a 1-block. It remains to show the commutativity $\hat{A}_1\hat{E}E_1'=\hat{A}_1A_1'E_1'$. In view of the projection $E_2\oplus E_2'\rightarrow\hat{A}_1$ in $X_2'\frown X_2$, this commutativity follows from $E_2\hat{A}_1\hat{E}E_1'=E_2\hat{E}E_1'=0=E_2\hat{A}_1A_1'E_1'$ and $E_2'\hat{A}_1\hat{E}E_1'=E_2'\hat{E}E_1'=E_2'A_1'E_1'=E_2'\hat{A}_1A_1'E_1'$.

The first part of the lemma shows in particular that upto equivalence the

2-block $(X'\wedge X)$ is uniquely determined by X and X'. The 1-blocks $X_2' \sqcap X_2$: $B \to E_2\oplus E_2' \to \hat{A}_1$, $\overline{W}$: $\hat{A}_1 \to \hat{E} \to A$ thus determined by W will be denoted by $W(1\oplus 2)$, $W(1\times 2)$ respectively. Also we denote by $W(1\bigcirc 2)$ the special exact sequence

$$W(1\bigcirc 2) = W(1\oplus 2)\bigcirc W(1\times 2):\ \emptyset \to B \to E_2\oplus E_2' \to \hat{E} \to A \to \emptyset.$$

Note that we have the following 2-blocks:

$$\begin{tikzcd}
 & \emptyset \arrow[r] \arrow[d] & E_2' \arrow[r, Rightarrow] \arrow[d, "c"] & E_2' \arrow[d] \\
W(1\oplus 2): & B \arrow[r] \arrow[d, Rightarrow] & E_2\oplus E_2' \arrow[r] \arrow[d, "\mathrm{ɔ}"] & \hat{A}_1 \arrow[d, "\hat{\varphi}_1"] \\
W(*,+): & B \arrow[r] & E_2 \arrow[r] & A_1\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
 & E_2' \arrow[r, Rightarrow] \arrow[d] & E_2' \arrow[r] \arrow[d] & \emptyset \arrow[d] \\
W(1\times 2): & \hat{A}_1 \arrow[r] \arrow[d, "\hat{\varphi}_1"] & \hat{E} \arrow[r] \arrow[d] & A \arrow[d, Rightarrow] \\
W(-,*): & A_1 \arrow[r] & E_1 \arrow[r] & A\,,
\end{tikzcd}$$

(5.2.1)

$$\begin{tikzcd}
 & \emptyset \arrow[r] \arrow[d] & E_2 \arrow[r, Rightarrow] \arrow[d, "c"] & E_2 \arrow[d] \\
W(1\oplus 2): & B \arrow[r] \arrow[d, "-e"] & E_2\oplus E_2' \arrow[r] \arrow[d, "\mathrm{ɔ}"] & \hat{A}_1 \arrow[d, "\hat{\varphi}_1'"] \\
W(+,*): & B \arrow[r] & E_2' \arrow[r] & A_1'\,,
\end{tikzcd}
\qquad
\begin{tikzcd}
 & E_2 \arrow[r, Rightarrow] \arrow[d] & E_2 \arrow[r] \arrow[d] & \emptyset \arrow[d] \\
W(1\times 2): & \hat{A}_1 \arrow[r] \arrow[d, "\hat{\varphi}_1'"] & \hat{E} \arrow[r] \arrow[d] & A \arrow[d, Rightarrow] \\
W(*,-): & A_1' \arrow[r] & E_1' \arrow[r] & A\,.
\end{tikzcd}$$

For a special exacts equence X: $\emptyset \to B \xrightarrow{\varphi} E \to \cdots$ we shall denote by $-X$ the special exact sequence obtained from X by changing the sign of the first map as $-X$: $\emptyset \to B \xrightarrow{-\varphi} E \to \cdots$. Then from the above 2-blocks we get the following result:

PROPOSITION 5.1. *Let W be a 2-block. There exist special projections in $\mathcal{S}\mathcal{E}^2\mathcal{A}$ giving $W(1\bigcirc 2) \simeq W(1,2)$ and $W(1\bigcirc 2) \simeq -W(2,1)$. In particular we have*

$$[W(1,2)] + [W(2,1)] = 0.$$

For 1-blocks X: $B \to E \to A$, X': $B \to E' \to A'$ and elements $m \in M(A)$, $m' \in M(A')$ we shall write

$$(X, m) \boxplus (X', m')$$

if in the 2-block $(X'\wedge X)$ there exists an element $l \in M(\hat{A})$ such that

$$\hat{\varphi}\circ l = m, \qquad \hat{\varphi}'\circ l = m'.$$

Also for two special exact sequences $X_2\bigcirc X_1$: $\emptyset \to B \to E_2 \to E_1 \to A \to \emptyset$, $X_2'\bigcirc X_1'$: $\emptyset \to B \to E_2' \to E_1' \to A \to \emptyset$ we shall write

$$(X_2\bigcirc X_1) \boxplus (X_2'\bigcirc X_1')$$

if there exists a 2-block W such that $W(1,2) = X_2\bigcirc X_1$, $W(2,1) = X_2'\bigcirc X_1'$.

PROPOSITION 5.2. $(X_2\bigcirc X_1) \boxplus (X_2'\bigcirc X_1')$ *if and only if* $(X_2, [X_1]) \boxplus (X_2', [X_1'])$.

PROOF. Suppose the 2-block W gives $(X_2\bigcirc X_1) \boxplus (X_2'\bigcirc X_1')$. Because of the maps $W(1\times 2) \to W(-,*)$, $W(1\times 2) \to W(*,-)$ displayed earlier, we have $[W(1\times 2)]\circ\hat{\varphi}_1 = [W(-,*)] = [X_1]$ and $[W(1\times 2)]\circ\varphi_1' = [W(*,-)] = [X_1']$. Hence the element

$[W(1\times 2)]\in \mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A, \hat{A}_1)$ gives $(X_2, [X_1])\boxplus(X_2', [X_1'])$. Conversely suppose $\hat{X}_1$: $\hat{A}_1\to\hat{E}\to A$ is a 1-block such that $[\hat{X}_1]\in \mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A, \hat{A}_1)$ gives $(X_2, [X_1])\boxplus(X_2', [X_1'])$. This means $\hat{\varphi}_1\circ[\hat{X}_1]=[X_1]$, $\hat{\varphi}_1'\circ[\hat{X}_1]=[X_1']$, and so we have commutative diagrams

$$\begin{array}{ccccccc} \hat{X}_1: & \hat{A}_1 & \to & \hat{E} & \to & A \\ & \downarrow{\scriptstyle \hat{\varphi}_1} & & \downarrow & & \| \\ X_1: & A_1 & \to & E_1 & \to & A, \end{array} \qquad \begin{array}{ccccccc} \hat{X}_1: & \hat{A}_1 & \to & \hat{E} & \to & A \\ & \downarrow{\scriptstyle \hat{\varphi}_1'} & & \downarrow & & \| \\ X_1': & A_1' & \to & E_1' & \to & A. \end{array}$$

Define $E_2\to\hat{E}$, $E_2'\to\hat{E}$ by $E_2\hat{E}=E_2\hat{A}_1\hat{E}$, $E_2'\hat{E}=E_2'\hat{A}_1\hat{E}$. Then by Lemma 3×3.3 (iv*) the sequences $E_2\to\hat{E}\to E_1'$, $E_2'\to\hat{E}\to E_1$ are 1-blocks. This reconstructs a 2-block W giving $(X_2\bigcirc X_1)\boxplus(X_1'\bigcirc X_2')$.

THEOREM $\boxplus$ (ANTICOMMUTATIVITY THEOREM).

(i) *For 1-blocks* $X: B\to E\to A$, $X': B\to E'\to A'$ *and* $m\in M(A)$, $m'\in M(A')$ *we have* $(X, m)\boxplus(X', m')$ *if and only if* $[X]\circ m+[X']\circ m'=0\in \mathcal{S}^1M(B)$.

(ii) *For special exact sequences* $X, X'\in \mathrm{EXT}^2{}_{(\mathcal{A},\mathcal{S})}(A, B)$ *we have* $X\boxplus X'$ *if and only if* $[X]+[X']=0$.

PROOF. In virtue of Proposition 5.2, (ii) is a special case of (i). So we shall prove (i). Suppose in the 2-block $(X'\wedge X)$ an element $l\in M(\hat{A})$ gives $(X, m)\boxplus(X', m')$, i.e., $\hat{\varphi}\circ l=m$, $\hat{\varphi}'\circ l=m'$. Then because of $[X]\circ\hat{\varphi}=[X'\urcorner X]=-[X']\circ\hat{\varphi}'$, we get $[X]\circ m+[X']\circ m'=[X]\circ\hat{\varphi}\circ l+[X']\circ\hat{\varphi}'\circ l=[X'\urcorner X]\circ l-[X'\urcorner X]\circ l=0$. Conversely suppose $[X]\circ m+[X']\circ m'=0$. Recall that with regard to the 1-block $X'\frown X$: $B\to\hat{A}\xrightarrow{\tilde{\varphi}}A\oplus A'$ and the canonical injections $A\xrightarrow{c_1}A\oplus A'\xleftarrow{c_2}A'$ we have $[X'\frown X]\circ c_1=[X]$, $[X'\frown X]\circ c_2=[X']$. Thus we now have $[X'\frown X]\circ(c_1\circ m+c_2\circ m')=0$, and so by Proposition 4.5 there exists an element $l\in M(\hat{A})$ such that $\tilde{\varphi}\circ l=c_1\circ m+c_2\circ m'$. On the other hand as to the canonical projections $A\xleftarrow{\text{ɔ}_1}A\oplus A'\xrightarrow{\text{ɔ}_2}A'$ we have $\text{ɔ}_1\circ\tilde{\varphi}=\hat{\varphi}$, $\text{ɔ}_2\circ\tilde{\varphi}=\hat{\varphi}'$. Hence we obtain $\hat{\varphi}\circ l=\text{ɔ}_1\circ(c_1\circ m+c_2\circ m')=m$, $\hat{\varphi}'\circ l=\text{ɔ}_2\circ(c_1\circ m+c_2\circ m)=m'$. This shows $(X, m)\boxplus(X', m')$.

SIMILARITY THEOREM I_2. *For two special exact sequences*

$$\begin{array}{l} X:\ \emptyset\longrightarrow B\longrightarrow E_2\longrightarrow E_1\longrightarrow A\longrightarrow\emptyset, \\ X':\ \emptyset\longrightarrow B\longrightarrow E_1'\longrightarrow E_1'\longrightarrow A\longrightarrow\emptyset \end{array}$$

the following conditions are equivalent to each other:

(1) $X\sim X'$

(2) *There exists a special exact sequence* $\hat{X}\in \mathrm{EXT}^2{}_{(\mathcal{A},\mathcal{S})}(A,B)$ *such that* $X\underset{\leftarrow}{\simeq}\hat{X}\underset{\rightarrow}{\simeq}X'$;

(2*) *There exists a special exact sequence* $\check{X}\in \mathrm{EXT}^2{}_{(\mathcal{A},\mathcal{S})}(A,B)$ *such that* $X\underset{\rightarrow}{\simeq}\check{X}\underset{\leftarrow}{\simeq}X'$;

(3) *There exists a commutative diagram*

$$\begin{tikzcd}
X: & \emptyset \arrow[r] & B \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r] & A \arrow[r] & \emptyset \\
\hat{X}: & \emptyset \arrow[r] & B \arrow[r] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & E_2 \oplus E_2 \arrow[r] \arrow[u, "\supset"'] \arrow[d, "\supset"'] & \hat{E}_1 \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, Rightarrow] \arrow[d, Rightarrow] & \emptyset \\
X': & \emptyset \arrow[r] & B \arrow[r] & E_2' \arrow[r] & E_1' \arrow[r] & A \arrow[r] & \emptyset,
\end{tikzcd}$$

where $\hat{X}$ *is special exact, and* $\hat{E}_1 E_1$, $\hat{E}_1 E_1'$ *are special projections with*

$$E_2' \subset (E_2 \oplus E_2') \hat{E}_1 = \text{kernel } \hat{E}_1 E_1, \quad E_2 \subset (E_2 \oplus E_2') \hat{E}_1 = \text{kernel } \hat{E}_1 E_1';$$

(3*) *There exists a commutative diagram*

$$\begin{tikzcd}
X: & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\subset"] & A \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
\check{X}: & \emptyset \arrow[r] & B \arrow[r] & \check{E}_2 \arrow[r] & E_1 \oplus E_1' \arrow[r] & A \arrow[r] & \emptyset \\
X': & \emptyset \arrow[r] & B \arrow[r] \arrow[u, Rightarrow] & E_2' \arrow[r] \arrow[u] & E_1' \arrow[r] \arrow[u, "\subset"'] & A \arrow[r] \arrow[u, Rightarrow] & \emptyset,
\end{tikzcd}$$

where $\check{X}$ *is special exact, and* $E_2 \check{E}_2$, $E_2' \check{E}_2$ *are special injections with*

$$\check{E}_2 (E_1 \oplus E_1') \supset E_1' = \text{cokernel } E_2 \check{E}_2, \quad \check{E}_2 (E_1 \oplus E_1') \supset E_1 = \text{cokernel } E_1' \check{E}_2 .$$

PROOF. The implications (3)→(2)→(1), (3*)→(2*)→(1) are obvious. In view of duality principle it will suffice to show the implication (1) →(3). As before $-X'$ will denote the sequence X' with $B \xrightarrow{\psi'} E_2'$ replaced by $B \xrightarrow{-\psi'} E_2'$. Suppose (1) $X \sim X'$. Then we have $[X] + [-X'] = 0$, and so by Theorem ⊞ there exists a 2-block

$$\begin{tikzcd}
& B \arrow[r] \arrow[d, "-\psi'"] & E_2 \arrow[r] \arrow[d] & A_1 \arrow[d] \\
W_1: & E_2' \arrow[r] \arrow[d] & \hat{E}_1 \arrow[r] \arrow[d] & E_1 \arrow[d] \\
& A_1' \arrow[r] & E_1' \arrow[r] & A
\end{tikzcd}$$

such that $W_1(1, 2) = X$, $W_1(2, 1) = -X'$. Thus taking into account the four 2-blocks as in (5.2.1), we have only to put $\hat{X} = W_1(1 \bigcirc 2)$ to obtain (3).

5.3. Let

$$X: \ \emptyset \longrightarrow B \longrightarrow E_2 \longrightarrow E_1 \longrightarrow A \longrightarrow \emptyset,$$
$$X': \ \emptyset \longrightarrow B \longrightarrow E_2' \longrightarrow E_1' \longrightarrow A' \longrightarrow \emptyset$$

be special exact sequences. For elements $m \in M(A)$, $m' \in M(A')$ we shall write

$$(X, m) ⊞_2 (X', m')$$

if there exist 1-blocks U: $B \to \tilde{E}_2 \to \tilde{A}_1$, V: $\tilde{A}_1 \to \tilde{E}_1 \to A$, V': $\tilde{A}_1 \to \tilde{E}_1' \to A'$ such that

$$(U \bigcirc V) ⊞ X, \ (U \bigcirc V') ⊞ X', \ (V, m) ⊞ (V', m').$$

In virtue of Theorem ⊞ these three relations are respectively equivalent to

$$[U]\circ[V]+[X]=0,\ [U]\circ[V']+[X']=0,\ [V]\circ m+[V']\circ m'=0.$$

Therefore we must have $[X]\circ m+[X']\circ m'=0$. We now assert the converse, namely the following:

PROPOSITION 5.3. *$[X]\circ m+[X']\circ m'$ implies $(X, m)\boxplus_2(X', m')$.*

PROOF. We decompose X, X' into 1-blocks as $X=X_2\bigcirc X_1$, $X'=X_1'\bigcirc X_1'$. Firstly we have

$$[-X_2']\circ[X_1']+[X']=0,\ [X_1']\circ m+[X_1']\circ(-m)=0,$$

and so setting $U=-X_2'$, $V=V'=X_1'$ we obtain

$$(X', -m')\boxplus_2(X', m').$$

On the other hand we have $[X]\circ m=-[X']\circ m'=[X']\circ(-m')$. Therefore it will suffice to show that in the situation $[X'']\in\mathrm{Ext}^2{}_{(\mathcal{A},\mathcal{S})}(A'', B)$, $m''\in M(A'')$, $[X]\circ\alpha=[X'']$, $\alpha\circ m''=m$ $(\alpha: A''\to A)$, we have

$$(X'', m'')\boxplus_2(X', m')\ \textit{if and only if}\ (X, m)\boxplus_2(X', m').$$

We set $X_1\bigcirc\alpha=X_1{}^\alpha$. Then we get $[X'']=[X_2\bigcirc X_1{}^\alpha]$, and so $(U\bigcirc V)\boxplus X''$ if and only if $(U\bigcirc V)\boxplus(X_2\bigcirc X_1{}^\alpha)$. Thus we have

$$(X'', m'')\boxplus_2(X', m')\ \textit{if and only if}\ (X_2\bigcirc X_1{}^\alpha, m'')\boxplus_2(X', m').$$

Therefore our task is reduced to proving that

$$(X_2\bigcirc X_1{}^\alpha, m'')\boxplus_2(X', m')\ \textit{if and only if}\ (X_2\bigcirc X_1, m)\boxplus_2(X', m'). \tag{5.3.1}$$

As for the 'if' part of (5.3.1) let U, V, V' be the 1-blocks giving $(X_2\bigcirc X_1, m)\boxplus_2(X', m')$, i.e., $(U\bigcirc V)\boxplus(X_2\bigcirc X_1)$, $(U\bigcirc V')\boxplus X'$, and $(V, m)\boxplus(V', m')$. Setting $V^\alpha=V\bigcirc\alpha$, we obtain $[U]\circ[V^\alpha]+[X_2]\circ[X_1{}^\alpha]=([U]\circ[V]+[X_2]\circ[X_1])\circ\alpha=0$, $[V^\alpha]\circ m''+[V']\circ m'=[V]\circ\alpha\circ m''+[V']\circ m'=[V]\circ m+[V']\circ m'=0$. Thus the 1-blocks U, V^α, V' give the required relation $(X_2\bigcirc X_1{}^\alpha, m'')\boxplus_2(X', m')$. Finally for the 'only if' part of (5.3.1) suppose $(X_2\bigcirc X_1{}^\alpha, m'')\boxplus_2(X', m')$. Because of $[-X_1]\circ m+[X_1{}^\alpha]\circ m''=-[X_1]\circ m+[X_1]\circ\alpha\circ m''=0$ we have, with regard to the 2-block

$$((-X_1)\wedge X_1{}^\alpha):\quad \begin{tikzcd} A \arrow[r] \arrow[d, "-\psi_1"] & E_1'' \arrow[r] \arrow[d] & A'' \arrow[d, equal] \\ E_1 \arrow[r] \arrow[d] & \hat{A} \arrow[r, "\hat{\varphi}''"] \arrow[d, "\hat{\varphi}"] & A'' \arrow[d] \\ A \arrow[r, Rightarrow] & A \arrow[r] & \emptyset, \end{tikzcd}$$

an element $l\in M(\hat{A})$ such that $\hat{\varphi}\circ l=m$, $\hat{\varphi}''\circ l=m''$. Also we have special projections of 1-blocks

$$\begin{tikzcd} (-X_1)\sqcap X_1{}^\alpha \arrow[d] & : & A_1 \arrow[r] \arrow[d, equal] & E_1''\oplus E_1 \arrow[r] \arrow[d, "\partial"] & \hat{A} \arrow[d, "\hat{\varphi}''"] \\ X_1{}^\alpha & : & A_1 \arrow[r] & E'' \arrow[r] & A'', \end{tikzcd} \qquad \begin{tikzcd} (-X_1)\sqcap X_1{}^\alpha \arrow[d] & : & A_1 \arrow[r] \arrow[d, equal] & E_1''\oplus E_1 \arrow[r] \arrow[d] & \hat{A} \arrow[d, "\hat{\varphi}"] \\ X_1 & : & A_1 \arrow[r] & E_1 \arrow[r] & A. \end{tikzcd}$$

By a suitable substitution in the 'if' part of (5.3.1) we obtain $(X_2\bigcirc((-X_1)\sqcap X_1^\alpha), l)$ $\boxplus_2(X', m')$ from $(X_2\bigcirc X_1^\alpha, m'')\boxplus_2(X', m')$. This shows that without loss of generality we may assume the map of 1-blocks

$$\begin{tikzcd}
X_1^\alpha\colon \arrow[d] & A_1 \arrow[r] \arrow[d, Rightarrow, no head] & E_1'' \arrow[r] \arrow[d] & A'' \arrow[d, "\alpha"] \\
X\ \colon & A_1 \arrow[r] & E \arrow[r] & A
\end{tikzcd}$$

is a special projection. Assuming so, let $\tilde{U}$, $\tilde{V}$, $\tilde{V}'$ be the 1-blocks giving $(X_2\bigcirc X_1^\alpha,$ $m'')\boxplus_2(X', m')$, i.e., $(\tilde{U}\bigcirc\tilde{V})\boxplus(X_2\bigcirc X_1^\alpha)$, $(\tilde{U}\bigcirc\tilde{V})\boxplus X'$, $(\tilde{V}, m'')\boxplus(\tilde{V}', m')$. Further let

$$\begin{tikzcd}
 & B \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & A_1 \arrow[d] \\
W''\colon & \tilde{E}_2'' \arrow[r] \arrow[d] & \hat{E} \arrow[r] \arrow[d] & E_1'' \arrow[d] \\
 & \tilde{A}_1'' \arrow[r] & \tilde{E}_1 \arrow[r] & A''
\end{tikzcd}$$

be the 2-block giving $(\tilde{U}\bigcirc\tilde{V})\boxplus(X_2\bigcirc X_1^\alpha)$. We consider W'' as a 1-block $\tilde{U}$ $\to W''(0, *)\to X_1^\alpha$ in $\mathcal{SE}^1\mathcal{A}$, and define $W''(0, *)\to X_1$ by $W''(0, *)X_1=W''(0, *)X_1^\alpha X_1$. This map of 1-blocks being a special projection, we take its kernel $U\to W''(0, *)$, and obtain the map of 2-blocks

$$\begin{tikzcd}
W''\colon \arrow[d] & \tilde{U} \arrow[r] \arrow[d] & W''(0, *) \arrow[r] \arrow[d, Rightarrow, no head] & X_1^\alpha \arrow[d] \\
W\ \colon & U \arrow[r] & W''(0, *) \arrow[r] & X_1\ .
\end{tikzcd}$$

Since the map $X_1^\alpha X_1$ has the identity on A_1, we may assume $W(*, +)=X_2$. Then as parts of the map $W''W$ we get the following maps of 1-blocks:

$$\begin{tikzcd}
\tilde{U}=W''(+, *)\colon \arrow[d] & B \arrow[r] \arrow[d, Rightarrow, no head] & \tilde{E}_2'' \arrow[r] \arrow[d] & \tilde{A}_1'' \arrow[d, "\alpha_1"] \\
U=W\ (+, *)\colon & B \arrow[r] & \tilde{E}_2 \arrow[r] & \tilde{A}_1\ ,
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{V}=W''(*, -)\colon \arrow[d] & \tilde{A}_1'' \arrow[r] \arrow[d, "\alpha_1"] & \tilde{E}_1 \arrow[r] \arrow[d, Rightarrow, no head] & A'' \arrow[d, "\alpha"] \\
V=W\ (*, -)\colon & \tilde{A}_1 \arrow[r] & \tilde{E}_1 \arrow[r] & A\ .
\end{tikzcd}$$

From these commutative diagrams we get $[U]\circ\alpha_1=[\tilde{U}]$, $[V]\circ\alpha=\alpha_1\circ[\tilde{V}]$. Then in setting $V'=\alpha_1\bigcirc\tilde{V}'$, we obtain

$$[U]\circ[V']+[X']=[U]\circ\alpha_1\circ[\tilde{V}']+[X']=[\tilde{U}]\circ[\tilde{V}']+[X']=0,$$
$$[V]\circ m+[V']\circ m'=[V]\circ\alpha\circ m''+\alpha_1\circ[\tilde{V}']\circ m'=\alpha_1\circ([\tilde{V}]\circ m''+[\tilde{V}']\circ m')=0.$$

In addition to these relations the 2-block W gives $(U\bigcirc V)\boxplus(X_2\bigcirc X_1)$. Thus the 1-blocks U, V, V' give the required relation $(X_2\bigcirc X_1, m)\boxplus_2(X', m')$, and the proposition is established.

5.4. Let

$$X\colon\ \emptyset\to B\to E_n\to E_{n-1}\to\cdots\to E_1\to A\to\emptyset$$

$$X' : \quad \emptyset \to B \to E'_n \to E'_{n-1} \to \cdots \to E'_1 \to A' \to \emptyset$$

be spəcial exact sequences ($n \geqq 3$). We decompose X, X' into 1-blocks as $X = X_n \bigcirc X_{n-1} \bigcirc \cdots \bigcirc X_1$, $X' = X'_n \bigcirc X'_{n-1} \bigcirc \cdots \bigcirc X'_1$. For elements $m \in M(A)$, $m' \in M(A')$ we define the relation

$$(X, m) \boxplus_n (X', m')$$

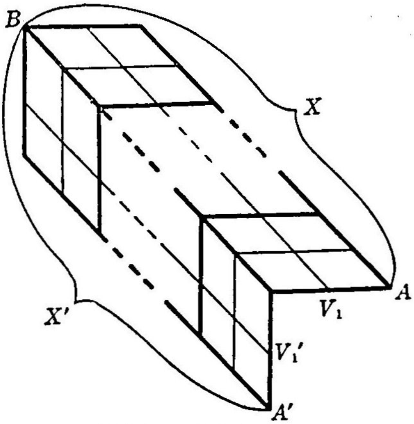
inductively as follows. Namely we write $(X, m) \boxplus_n (X', m')$ if there exist 1-blocks U_n, V_{n-1}, V'_{n-1} such that

$$(U_n \bigcirc V_{n-1}) \boxplus (X_n \bigcirc X_{n-1}), \quad (U_n \bigcirc V'_{n-1}) \boxplus (X'_n \bigcirc X'_{n-1}),$$
$$(V_{n-1} \bigcirc X_{n-2} \bigcirc \cdots \bigcirc X_1, m) \boxplus_{n-1} (V'_{n-1} \bigcirc X'_{n-2} \bigcirc \cdots \bigcirc X'_1, m').$$

In other words $(X, m) \boxplus_n (X', m')$ means existence of 1-blocks $U_n, \cdots, U_2, V_{n-1}, \cdots, V_1, V'_{n-1}, \cdots, V'_1$ such that

$$(U_n \bigcirc V_{n-1}) \boxplus (X_n \bigcirc X_{n-1}), \ (U_n \bigcirc V'_{n-1}) \boxplus (X'_n \bigcirc X'_{n-1}),$$
$$(U_i \bigcirc V_{i-1}) \boxplus (V_i \bigcirc X_{i-1}), \ (U_i \bigcirc V'_{i-1}) \boxplus (V_i' \bigcirc X'_{i-1}) \qquad (2 \leqq i \leqq n-1),$$
$$(V_1, m) \boxplus (V'_1, m').$$

Schematically this means existence of the following patch work of 2-blocks:



$([V_1] \circ m + [V_1'] \circ m' = 0)$.

Since each of these $2n-1$ $\boxplus$'s gives anticommutativity,

$$(X, m) \boxplus_n (X', m') \text{ implies } [X] \circ m + [X'] \circ m' = 0.$$

THEOREM $\boxplus_n$. *$(X, m) \boxplus_n (X', m')$ if and only if $[X] \circ m + [X'] \circ m' = 0$ ($n \geqq 2$).*

PROOF. We have only to show the 'if' part. We prove it by induction on n. Firstly Theorem $\boxplus_2$ is true in virtue of Proposition 5.3. For $n \geqq 3$ suppose $[X] \circ m + [X'] \circ m' = 0$. Then we have

$$[X_n \bigcirc X_{n-1}]\circ([X_{n-2}\bigcirc\cdots\bigcirc X_1]\circ m)+[X'_n\bigcirc X'_{n-1}]\circ([X'_{n-2}\bigcirc\cdots\bigcirc X'_1]\circ m')=0,$$

and so by Theorem $\boxplus_2$ there exist 1-blocks U_n, V_{n-1}, V'_{n-1} such that

$$(U_n\bigcirc V_{n-1})\boxplus(X_n\bigcirc X_{n-1}),\ (U_n\bigcirc V'_{n-1})\boxplus(X'_n\bigcirc X'_{n-1}),$$
$$(V_{n-1}, [X_{n-2}\bigcirc\cdots\bigcirc X_1]\circ m)\boxplus(V'_{n-1}, [X'_{n-2}\bigcirc\cdots\bigcirc X'_1]\circ m').$$

The last relation implies

$$[V_{n-1}\bigcirc X_{n-2}\bigcirc\cdots\bigcirc X_1]\circ m+[V'_{n-1}\bigcirc X'_{n-2}\bigcirc\cdots\bigcirc X'_1]\circ m'.$$

Therefore by induction hypothesis we get

$$(V_{n-1}\bigcirc X_{n-2}\bigcirc\cdots\bigcirc X_1, m)\boxplus_{n-1}(V'_{n-1}\bigcirc X'_{n-2}\bigcirc\cdots\bigcirc X'_1, m'),$$

and so $(X, m)\boxplus(X', m')$, completing the proof.

Now suppose $[X]\circ m=[X']\circ m'$. Then $[X]\circ m+[-X']\circ m'=0$, and so (X, m) $\boxplus_n(-X', m')$. Therefore there exists a patch work as displayed above with X' replaced by $-X'$. We consider it as consisting of two 1-blocks in $(\mathcal{S}\mathcal{E}^{n-1}\mathcal{A}, \mathcal{S}^{(n-1)})$,

$$W:\ U_n\bigcirc\cdots\bigcirc U_2 \longrightarrow V \longrightarrow X_{n-1}\bigcirc\cdots\bigcirc X_1,$$
$$W':\ U_n\bigcirc\cdots\bigcirc U_2 \longrightarrow V' \longrightarrow X'_{n-1}\bigcirc\cdots\bigcirc X'_1.$$

These two 1-blocks W, W' yield the 2-block

$$(W'\wedge W):\quad \begin{array}{ccccc} U_n\bigcirc\cdots\bigcirc U_2 & \longrightarrow & V & \longrightarrow & X_{n-1}\bigcirc\cdots\bigcirc X_1 \\ \downarrow & & \downarrow & & \Downarrow \\ V' & \longrightarrow & \hat{V} & \longrightarrow & X_{n-1}\bigcirc\cdots\bigcirc X_1 \\ \downarrow & & \downarrow & & \downarrow \\ X'_{n-1}\bigcirc\cdots\bigcirc X'_1 & \Rightarrow & X'_{n-1}\bigcirc\cdots\bigcirc X'_1 & \longrightarrow & \emptyset \end{array}$$

in $(\mathcal{S}\mathcal{E}^{n-1}\mathcal{A}, \mathcal{S}^{(n-1)})$. Take it as a special exact sequence of 2-blocks in $(\mathcal{A}, \mathcal{S})$. It starts from the 2-block

$$(-X'_n\wedge X_n):\quad \begin{array}{ccccc} B & \longrightarrow & E_n & \longrightarrow & A_{n-1} \\ \downarrow - & & \downarrow & & \Downarrow \\ E'_n & \longrightarrow & \hat{A}_{n-1} & \longrightarrow & A_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ A'_{n-1} & \Rightarrow & A'_{n-1} & \longrightarrow & \emptyset, \end{array}$$

and ends at the 2-block

$$\begin{array}{cccccc} & V'_1: & & & & \\ V_1: & \tilde{A}_1 & \longrightarrow & F_1 & \longrightarrow & A \\ & \downarrow & & \downarrow & & \Downarrow \\ (V'_1\wedge V_1): & F'_1 & \longrightarrow & \hat{A}_1 & \longrightarrow & A \\ & \downarrow & & \downarrow & & \downarrow \\ & A' & \Longrightarrow & A' & \longrightarrow & \emptyset\,. \end{array}$$

We now join the two special projections of 1-blocks

$$\begin{tikzcd}
X_n \arrow[d, leftarrow] & : & B \arrow[r] \arrow[d, Leftarrow] & E_n \arrow[r] \arrow[d, leftarrow, "\partial"] & A_{n-1} \arrow[d, leftarrow] \\
-X'_n \sqcap X_n \arrow[d] & : & B \arrow[r] \arrow[d, Rightarrow] & E_n \oplus E'_n \arrow[r] \arrow[d, "\partial"] & \hat{A}_{n-1} \arrow[d] \\
X'_n & : & B \arrow[r] & E'_n \arrow[r] & A'_{n-1}
\end{tikzcd}$$

with the special projections,

$$\begin{tikzcd}
X_{n-1}\bigcirc\cdots\bigcirc X_1 \arrow[d, leftarrow] & : & \emptyset \arrow[r] & A_{n-1} \arrow[r] \arrow[d, leftarrow] & E_{n-1} \arrow[r] \arrow[d, leftarrow] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d, leftarrow] & A \arrow[r] \arrow[d, leftarrow, "\hat{\varphi}"] & \emptyset \\
\hat{V} \arrow[d] & : & \emptyset \arrow[r] & \hat{A}_{n-1} \arrow[r] \arrow[d] & \hat{E}_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{E}_1 \arrow[r] \arrow[d] & \hat{A} \arrow[r] \arrow[d, "\hat{\varphi}'"] & \emptyset \\
X'_{n-1}\bigcirc\cdots\bigcirc X'_1 & : & \emptyset \arrow[r] & A'_{n-1} \arrow[r] & E'_{n-1} \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r] & A' \arrow[r] & \emptyset
\end{tikzcd}$$

included in the 2-block $(W'\wedge W)$. Then, in setting $\check{X}=(-X'_n \sqcap X_n)\bigcirc\hat{V}$, we obtain two special projections in $(\mathcal{S}\mathcal{E}^n\mathcal{A}, \mathcal{S}^{(n)})$ of the form

$$\begin{tikzcd}
X \arrow[d, leftarrow] & : & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Leftarrow] & E_n \arrow[r] \arrow[d, leftarrow, "\partial"] & E_{n-1} \arrow[r] \arrow[d, leftarrow] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d, leftarrow] & A \arrow[r] \arrow[d, leftarrow, "\hat{\varphi}"] & \emptyset \\
\hat{X} \arrow[d] & : & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & E_n\oplus E'_n \arrow[r] \arrow[d, "\partial"] & \hat{E}_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{E}_1 \arrow[r] \arrow[d] & \hat{A} \arrow[r] \arrow[d, "\hat{\varphi}'"] & \emptyset \\
X' & : & \emptyset \arrow[r] & B \arrow[r] & E'_n \arrow[r] & E'_{n-1} \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r] & A' \arrow[r] & \emptyset\ .
\end{tikzcd}$$

Because of the relation $(V_1, m)\boxplus(V'_1, m')$ there exists an element $l\in M(\hat{A})$ such that $\hat{\varphi}\circ l=m$, $\hat{\varphi}'\circ l=m'$. Thus in $\mathcal{S}^n M(B)$ $(n\geqq 1)$ we have $[X]\circ m=[X']\circ m'$ if and only if there exists a third element $[\hat{X}]\circ l$ and special projections $\xi: \hat{X}\to X$, $\xi': \hat{X}\to X'$ such that $\mathcal{S}E^n_+(\xi)=\mathcal{S}E^n_+(\xi')=e_B$, $\mathcal{S}E^n_-(\xi)\circ l=m$, $\mathcal{S}E^n_-(\xi')\circ l=m'$. In particular, setting $M(a)=\mathrm{Ext}^1{}_{(\mathcal{A},\mathcal{S})}(A, a)$, we get finally the following:

SIMILARITY THEOREM I_n. *For two special exact sequences*

$$X:\ \emptyset\longrightarrow B\longrightarrow E_n\longrightarrow\cdots\longrightarrow E_1\longrightarrow A\longrightarrow\emptyset,$$
$$X':\ \emptyset\longrightarrow B\longrightarrow E'_n\longrightarrow\cdots\longrightarrow E'_1\longrightarrow A\longrightarrow\emptyset$$

the following conditions are equivalent to each other :

(1) $X\sim X'$.

(2) *There exists* $\hat{X}\in\mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ *such that* $X\leftarrow\hat{X}\rightarrow X'$ (with $\simeq$).

(2*) *There exists* $\check{X}\in\mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{C})}(A, B)$ *such that* $X\rightarrow\check{X}\leftarrow X'$ (with $\simeq$).

(3) *There exists a commutative diagram*

$$\begin{tikzcd}
X \arrow[d, leftarrow] & : & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Leftarrow] & E_n \arrow[r] \arrow[d, leftarrow, "\partial"] & E_{n-1} \arrow[r] \arrow[d, leftarrow] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[d, leftarrow] & A \arrow[r] \arrow[d, Leftarrow] & \emptyset \\
\hat{X} \arrow[d] & : & \emptyset \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & E_n\oplus E'_n \arrow[r] \arrow[d, "\partial"] & \hat{E}_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{E}_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, Rightarrow] & \emptyset \\
X' & : & \emptyset \arrow[r] & B \arrow[r] & E'_n \arrow[r] & E'_{n-1} \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r] & A \arrow[r] & \emptyset\ .
\end{tikzcd}$$

where $\hat{X}$ is special exact and $\hat{X}\to X$, $\hat{X}\to X'$ are special projections in $(\mathcal{S}\mathcal{E}^n\mathcal{A},\ \mathcal{S}^{(n)})$.

(3*) *There exists a commutative diagram*

$$\begin{tikzcd}
X\colon \arrow[d] & \emptyset \arrow[r] & B \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d] & \cdots \arrow[r] & E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "c"] & A \arrow[r] \arrow[d, equal] & \emptyset \\
\check{X}\colon & \emptyset \arrow[r] & B \arrow[r] & \check{E}_n \arrow[r] & \cdots \arrow[r] & \check{E}_2 \arrow[r] & E_1\oplus E'_1 \arrow[r] & A \arrow[r] & \emptyset \\
X'\colon \arrow[u] & \emptyset \arrow[r] & B \arrow[r] \arrow[u, equal] & E'_n \arrow[r] \arrow[u] & \cdots \arrow[r] & E'_2 \arrow[r] \arrow[u] & E'_1 \arrow[r] \arrow[u, "c"] & A \arrow[r] \arrow[u, equal] & \emptyset,
\end{tikzcd}$$

where is $\check{X}$ special exact and $X\to\check{X}$, $X'\to\check{X}$ are special injections in $(\mathcal{S}\mathcal{E}^n\mathcal{A},\ \mathcal{S}^{(n)})$.

We note also that two similar exact sequences $X,\ X'\in \mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ $(n\geqq 2)$ are connected by $2n-3$ steps of modifications each consisting in a special projection from a third object $\hat{X}\in \mathrm{EXT}^n{}_{(\mathcal{A},\mathcal{S})}(A, B)$ of the following form $(1\leqq i\leqq n-1)$:

$$\begin{tikzcd}
X^{(k)}\colon & \emptyset \arrow[r] & B \arrow[r] & E_n \arrow[r] & \cdots \arrow[r] & E_{i+2} \arrow[r] & E_{i+1} \arrow[r] & E_i \arrow[r] & E_{i-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & A \arrow[r] & \emptyset \\
\hat{X}\colon & \emptyset \arrow[r] & B \arrow[r] \arrow[u, equal] \arrow[d, equal] & E_n \arrow[r] \arrow[u, equal] \arrow[d, equal] & \cdots \arrow[r] & E_{i+2} \arrow[r] \arrow[u, equal] \arrow[d, equal] & E_{i+1}\oplus E'_{i+1} \arrow[r] \arrow[u] \arrow[d, "\text{ɔ}"] & \hat{E}_i \arrow[r] \arrow[u] \arrow[d] & E_{i-1} \arrow[r] \arrow[u, equal] \arrow[d, equal] & \cdots \arrow[r] & E_1 \arrow[r] \arrow[u, equal] \arrow[d, equal] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & \emptyset \\
X^{(k+1)}\colon & \emptyset \arrow[r] & B \arrow[r] & E_n \arrow[r] & \cdots \arrow[r] & E_{i+2} \arrow[r] & E'_{i+1} \arrow[r] & E'_i \arrow[r] & E_{i-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & A \arrow[r] & \emptyset.
\end{tikzcd}$$

This is an immediate consequence of Theorem III_{n-1} and Similarity Theorem I_2.

As a corollary to Similarity Theorem I_n we get easily the following:

SIMILARITY THEOREM II. *For a special exact sequence*

$$X\colon\ \emptyset\longrightarrow B\longrightarrow E_n\longrightarrow\cdots\longrightarrow E_1\longrightarrow A\longrightarrow\emptyset \quad (n\geqq 2)$$

the following conditions are equivalent to each other:

(1) $[X]=0$;

(2) *There exists a commutative diagram*

$$\begin{tikzcd}
\hat{X}\colon & & \emptyset \arrow[r] & E_n \arrow[r] \arrow[d, equal] & \hat{E}_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{E}_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & \emptyset \\
X\colon & \emptyset \arrow[r] & B \arrow[r] & E_n \arrow[r] & E_{n-1} \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & A & \emptyset, \arrow[l]
\end{tikzcd}$$

where $\hat{X}$ is special exact:

(2*) *There exists a commutative diagram*

$$\begin{tikzcd}
X\colon & \emptyset \arrow[r] & B \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d] & \cdots \arrow[r] & E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, equal] & A \arrow[r] & \emptyset \\
\check{X}\colon & \emptyset \arrow[r] & B \arrow[r] & \check{E}_n \arrow[r] & \cdots \arrow[r] & \check{E}_2 \arrow[r] & E_1 \arrow[r] & \emptyset, &
\end{tikzcd}$$

where $\check{X}$ is special exact.

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