Valuations, the linear Artin approximation theorem and convergence of formal functions.

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§1. VALUATIONS.

Let Γ be an abelian group.

DEFINITION 1.1. *Γ is an* **ordered group** *if Γ may be written as a disjoint union*

$$\Gamma = P \coprod \{0\} \coprod (-P) \tag{1.1}$$

where P is a subset of Γ, closed under addition.

One should think of P as the set of "positive elements".

Remark 1.2. If Γ is an ordered group, (1) induces a total ordering on Γ, defined by

$$a < b \Longleftrightarrow b - a \in P.$$

In other words, an ordered group is a group with an ordering which respects addition.

Examples of ordered groups. $\mathbf{Z}$ with the usual ordering is an ordered group. There are infinitely many non-equivalent orderings which make $\mathbf{Z}^n$ into an ordered group. For example, we may take the *lexicographical ordering*, defined by

$$\begin{aligned} &(a_1, \dots, a_n) < (b_1, \dots, b_n) \quad \text{if there exists } i,\ 1 \le i \le n \ \text{ such that} \\ &\qquad a_j = b_j, \quad 1 \le j < i \qquad \text{and} \\ &\qquad a_i < b_i. \end{aligned} \tag{1.2}$$

Any additive subgroup of $\mathbf{R}$ with the usual ordering is an ordered group. More generally, let Γ_i, $1 \le i \le n$ be a collection of additive subgroups of $\mathbf{R}$. Then the direct sum

$$\Gamma := \bigoplus_{i=1}^{n} \Gamma_i \tag{1.3}$$

with lexicographical ordering (1.2) is an ordered group. All the ordered groups appearing in this paper will have the form (1.3).

DEFINITION 1.3. *Let Γ be an ordered group and Δ a subgroup. We say that Δ is an* **isolated subgroup** *if it is a segment in the ordering: if $a \in \Delta$ and $-a \le b \le a$ then $b \in \Delta$.*

For example, the only isolated subgroups of $\mathbf{Z}$ are (0) and $\mathbf{Z}$ itself. On the other hand, the ordered groups of (1.2) and (1.3) have exactly n non-zero isolated subgroups (counting the group itself).

Now let K be a field and Γ an ordered group. Let K^* denote the multiplicative group of K.

DEFINITION 1.4. *A* **valuation** ν *of* K *with value group* Γ *is a surjective group homomorphism*

$$\nu : K^* \to \Gamma$$

such that for any $x, y \in K$

$$\begin{aligned} \nu(x+y) &\geq \min(\nu(x), \nu(y)) && \text{and} \\ \nu(x+y) &= \min(\nu(x), \nu(y)) && \text{if } \nu(x) \neq \nu(y) \end{aligned}$$

(here and below we adopt the convention that $\nu(0) > \alpha$ *for any* $\alpha \in \Gamma$.

Example 1.5. Without doubt, the most important example of valuations are the **divisorial valuations.** Let X be an integral scheme of finite type over a field or a Dedekind domain. Let K denote the field of rational functions on X. Let D be an integral subscheme such that the local ring $\mathcal{O}_{X,D}$ is regular. Naturally associated to D is a valuation $\nu_D : K^* \to \mathbf{Z}$ defined by

$$\nu_D(f) = \mathrm{ord}_D(f),$$

where $\mathrm{ord}_D(f)$ denotes the order of zero or pole of f at the generic point of D. More precisely, if m denotes the maximal ideal of $\mathcal{O}_{X,D}$ and $f \in \mathcal{O}_{X,D}$ then

$$\nu(f) = \max\{n \mid f \in m^n\}.$$

If $f \notin \mathcal{O}_{X,D}$, write $f = \frac{a}{b}$, where $a, b \in \mathcal{O}_{X,D}$. Then

$$\nu(f) = \nu(a) - \nu(b).$$

Of course, this definition works for any D such that ord_D is additive on $\mathcal{O}_{X,D}$ (equivalently, the graded algebra $\oplus_{n=0}^{\infty} \frac{m^n}{m^{n+1}}$ is an integral domain). Valuations of this type are called divisorial (if D is not a divisor, we can always blow it up to make it into one). See Example 2.4 for a generalization of this definition to schemes which are not of finite type over K.

Note. The condition of being divisorial is stronger than saying that the value group is equal to $\mathbf{Z}$. Below we give an example of a non-divisorial valuation with value group $\mathbf{Z}$ (Example 2.7).

Example 1.6. More generally, given a flag of subschemes

$$D_n \subset D_{n-1} \subset \cdots \subset D_1 \subset D_0 = X$$

such that for each i, $1 \leq i \leq n$, ord_{D_i} is an additive function on $\mathcal{O}_{D_{i-1},D_i}$, there exists a unique valuation $\nu : K^* \to \mathbf{Z}^n$ (with lexicographical ordering) such that for any $f \in m_{X,D_i} \setminus m_{X,D_{i-1}}$ the first i entries in $\nu(f)$ are given by

$$\nu(f) = (\underbrace{0, 0, \ldots, 0}_{i-1 \text{ zeroes}}, \mathrm{ord}_{D_i} f, \ldots)$$

This valuation ν is defined using the concept of **composition of valuations** (see [20, §VI.10, p. 43]).

Let K be a field and $\nu : K^* \to \Gamma$ a valuation of ν. Associated with ν is a local subring R_ν of K:

$$R_\nu := \{x \in K \mid \nu(x) \geq 0\}.$$

The maximal ideal of R_ν is

$$m_\nu = \{x \in K \mid \nu(x) > 0\}.$$

We recall a well-known theorem from commutative algebra characterizing valuation rings.

THEOREM 1.7. *Let K be a field and R a local subring of K. The following four conditions are equivalent:*

(1) *R is the valuation ring R_ν for some valuation ν of K.*
(2) *For any $x \in K^*$, either $x \in R$ or $\frac{1}{x} \in R$ (or both).*
(3) *The set of ideals of R is totally ordered by inclusion.*
(4) *R is a maximal element in the set of local subrings of R with respect to the relation of domination (we say that a local ring (R_1, m_1)* ***dominates*** *(R_2, m_2), denoted $(R_1, m_1) > (R_2, m_2)$, if $R_2 \subset R_1$ and $m_2 = m_1 \cap R_2$).*

Remark 1.8. Let ν be a valuation of the field K with value group Γ. Then R_ν is noetherian if and only if $\Gamma = \mathbb{Z}$. Indeed, we have an injection ϕ of the set

$$\Gamma^+ := \{\alpha \in \Gamma \mid \alpha \geq 0\}$$

into the semigroup of ideals of R_ν under multiplication. Namely, if $\alpha \in \Gamma^+$, we define

$$\phi(\alpha) = q_\alpha := \{x \in K \mid \nu(x) \geq \alpha\}.$$

ϕ preserves the ordering and semigroup structure. (The complement of $\mathrm{Im}(\phi)$ in the set of ideals of R_ν is precisely the set of non-maximal, non-zero prime ideals of R_ν.) If R_ν is noetherian, then any descending chain of elements of Γ^+ must stabilize. In other words, Γ^+ is a well-ordered set. But the set of non-negative elements of an ordered group is well-ordered if and only if that group is equal to $\mathbb{Z}$ (otherwise we could take the minimal element among those for which the set of elements smaller than itself is infinite, subtract a non-zero element of Γ^+ and get a contradiction). Conversely, if $\Gamma = \mathbb{Z}$ then any ideal in R_ν is generated by a power of an element t such that $\nu(t) = 1$. In that case R_ν is a discrete valuation ring with regular parameter t.

We now digress to explain some of the motivation for studying valuations, coming from the problem of resolution of singularities. Indeed, let X be a reduced and irreducible projective variety over a field k and let K be the function field of X. Let ν be a valuation of K which vanishes on k. Say, $X \subset \mathbf{P}^n = \mathrm{Proj}\ k[x_0, \ldots, x_n]$ and assume that X is not contained in any hyperplane in $\mathbf{P}^n$. Then $\frac{x_i}{x_j}$ is a rational function on X. Renumbering the indices, if necessary, we may assume that $\nu(\frac{x_i}{x_j}) \geq 0$ whenever $i > j$. Let $U := X \cap k[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}]$. Then $\mathcal{O}_U \subset R_\nu$. Let $p := m_\nu \cap \mathcal{O}_U$. Then p is a well-defined (not necessarily closed) point on X. p is called the **center** of ν on X. p is the unique point in X such that $\mathcal{O}_{X,p} < R_\nu$. Thus a valuation gives a way of picking a point on every projective model of the field K.

Now condition (4) of Theorem 1.7 explains the geometric meaning of valuations. Let X, K and ν be as above. Consider the projective system of all the projective models $\{X_\alpha\}_{\alpha \in \Lambda}$ of K which admit a proper birational morphism to X. The maps in this projective system are birational proper morphisms $\pi_{\alpha\beta} : X_\alpha \to X_\beta$. Consider a collection of points $\{\xi_\alpha \in X_\alpha\}_{\alpha \in \Lambda}$ such that for any $\alpha, \beta \in \Lambda$

$$\pi_{\alpha\beta}(\xi_\alpha) = \xi_\beta.$$

Then we may consider the ring $R := \varinjlim_\alpha \mathcal{O}_{X_\alpha, \xi_\alpha}$. (4) of Theorem 1.7 implies that R is a valuation ring for some valuation ν. For each $\alpha \in \Lambda$, ξ_α is the center of ν in X_α. Thus giving a valuation ν which vanishes on k is equivalent to giving a collection of points $\xi_\alpha \in X_\alpha$, $\alpha \in \Lambda$, in a way which respects the maps $\pi_{\alpha\beta}$.

DEFINITION 1.9. *The* **abstract Riemann surface** *of the field K is defined to be the set of all the valuations of K vanishing on k.*

We denote the abstract Riemann surface of K by S. By the above discussion, we have, set-theoretically,

$$S = \varprojlim_\alpha X_\alpha.$$

We define the Zariski topology on S to be the inverse limit topology induced by all the Zariski topologies of the X_α. We have the following beautiful theorem of Zariski.

THEOREM 1.10. *[20, § VI.17, p. 113, Theorem 40]. S is compact.*

Note that this does not follow automatically from the compactness of each of the X_α because the Zariski topology is not Hausdorff.

Zariski's approach to the problem of resolution of singularities was to first prove the local uniformization theorem: for any projective variety X over k and any valuation ν of K, there exists a variety X', birational and proper over X such that the center of ν on X' is non-singular. Zariski proved this theorem for char $k = 0$ in all dimensions (the question is still open for char $k > 0$). Since regularity is an open condition, there exists then an open neighbourhood V of ν in S such that for any $\nu' \in V$ the center of ν' on X is also non-singular. By compactness, we can cover S by finitely many such open neighbourhoods. Then resolution of singularities reduces to the following problem. Consider two projective models X_1 and X_2 of K, birational and proper over X. Let V_1 and V_2 denote the preimages in S of the non-singular points of X_1 and X_2, respectively. The problem is to find a third projective model X_{12} dominating both X_1 and X_2 such that for any $\nu \in V_1 \cup V_2$, the center of ν in X_{12} is non-singular. Zariski solved this problem in dimensions two, and, with much greater difficulty, three. For higher dimensions, however, this problem seems to be almost as hard as the resolution of singularity itself. The ebbedded resolution in all dimensions (char $k = 0$) was later proved by Hironaka by an entirely different method. However, one may still hope that Zariski's valuation-theoretic approach, in some modified form will be useful for future research in resolution of singularities, particularly in the light of recent work on the formal path space of an algebraic variety [7].

§2. Inequalities on dimension and rank.

Let K be any field and $\nu : K^* \to \Gamma$ a valuation of K. We define the basic non-negative integer invariants associated with ν.

Definition 2.1. *The* **rank** *of ν is defined to be*

$$\operatorname{rk} \nu = \dim \ R_\nu,$$

the Krull dimension of R_ν. **Rational rank** *is*

$$\operatorname{rat.rk} \nu = \dim_{\mathbf{Q}} \Gamma \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Remark 2.2. rk ν equals the number of non-zero isolated subgroups of Γ (counting Γ itself). Indeed, there is a bijection between the set of isolated subgroups of Γ and the set of prime ideals of R given by

$$\Delta \leftrightarrow p = \{x \in R_\nu \mid \nu(x) \notin \Delta\}.$$

This bijection reverses inclusions.

It is easy to see that $rk\ \nu \leq \operatorname{rat.rk} \nu$. For instance, in Example 1.5

$$\operatorname{rk} \nu = \operatorname{rat.rk} \nu = 1.$$

In Example 1.6

$$\operatorname{rk} \nu = \operatorname{rat.rk} \nu = n.$$

In (1.3) we have $\operatorname{rk} \nu = n$ and $\operatorname{rat.rk} \nu = \sum_{i=1}^{n} \operatorname{rat.rk} \Gamma_i$.

We now fix the situation which we shall study for most of the paper. Let (R, m, k) be a local noetherian domain with field of fractions K. Let $\nu : K^* \to \Gamma$ be a valuation of K such that $R < R_\nu$. In this case there is another integer invariant we can associate to ν. Namely, the condition $R < R_\nu$ implies that we have a natural injection $k \hookrightarrow \frac{R_\nu}{m_\nu}$. We define

$$\operatorname{tr.deg}_k \nu := \operatorname{tr.deg}(\frac{R_\nu}{m_\nu}/k).$$

The starting point of this work is an old theorem of Abhyankar. Let $n = \dim\ R$.

THEOREM 2.3. *[1, Theorem 1, p. 330].*

(1)

$$\text{rat. rk } \nu + \text{tr. deg}_k \nu \leq \dim \ R. \tag{2.1}$$

(2) *If equality holds in (2.1) then $\Gamma \cong \mathbf{Z}^n$ and $\frac{R_\nu}{m_\nu}$ is finitely generated over k.*

(3) *If, moreover,*

$$\text{rk } \nu + \text{tr. deg}_k \nu = \dim \ R$$

then $\Gamma \cong \mathbf{Z}^n$ with the lexicographical ordering.

We gave a sketch of a simple proof of Abhyankar's Theorem in [15] (full details, as for most of the new results stated here, will appear in [17]). Also in [15] we give an example of a rank 2 valuation centered in a local non-noetherian domain of dimension 1, so that the noetherian hypothesis is necessary even for the weaker inequality (2.1) with rational rank replaced by rank.

Example 2.4. If ν is a divisorial valuation, then $\text{rk } \nu = \text{rat. rk } \nu = 1$ and $\text{tr. deg}_k \nu = n-1$ (the transcendence basis of $\frac{R_\nu}{m_\nu}$ over k is given by the transcendence basis of the field of rational functions on D in the case $\dim\ D = n-1$, i.e. when D is a divisor; blowing up D, if necessary, we can always reduce to this case). Hence for a divisorial valuation equalities hold in (1) and (3). We can now generalize our definition of divisorial valuations to mean any valuation centered in an n-dimensional local noetherian domain (R, m, k), having value group $\mathbf{Z}$ and such that $\text{tr. deg}_k \nu = n-1$.

Example 2.5. Let u, v be independent variables and let $R = k[u,v]_{(u,v)}$. Consider the valuation of $K = k(u,v)$ with value group $(1, \sqrt{2})$ (i.e. the additive subgroup of $\mathbf{R}$ generated by 1 and $\sqrt{2}$), defined by

$$\nu(v) = 1$$
$$\nu(u) = \sqrt{2}.$$

Then $\text{rk } \nu = 1$, $\text{rat. rk } \nu = 2$ and $\text{tr. deg}_k \nu = 0$, so that ν satisfies equality in (1) of Theorem 2.3, but not in (3).

Example 2.6. Let R and K be as in Example 2.5. Let $\Gamma = \mathbf{Z}^2$ in lexicographical ordering and let $\nu : K^* \to \Gamma$ be defined by

$$\nu(v) = (0,1)$$
$$\nu(u) = (1,0).$$

Then $\text{rk } \nu = \text{rat. rk } \nu = 2$ and $\text{tr. deg}_k \nu = 0$, so that ν satisfies the equality in both (1) and (3) of Theorem 2.3.

Note that in both Examples 2.5 and 2.6 ν is completely determined by $\nu(u)$ and $\nu(v)$. This is true precisely because in the cases considered $\nu(u)$ and $\nu(v)$

are linearly independent over $\mathbf{Z}$. In general, $\nu(u)$ and $\nu(v)$ determine the value of every monomial in $k[u,v]$. If $\nu(u)$ and $\nu(v)$ are integrally independent, then all the monomials have distinct values, so by the axioms of valuations the value of each polynomial equals the minimum value of its monomials. Then ν is completely determined on $k[u,v]$, hence on all of K. Next, we give two examples of valuations for which strict inequality holds in (1) of Theorem 2.3.

Example 2.7. Let R and K be as in the preceding Example. Let $\hat{R}$ denote the completion of R and $\hat{K}$ the field of fractions of $\hat{R}$. Write $\hat{R} = k[[u,v]]$. Consider a formal power series

$$t = u + \sum_{i=1}^{\infty} c_i v^i \in \hat{R} \setminus K, \quad \text{where} \quad c_i \in k^*.$$

Let $\hat{\Gamma} = \mathbf{Z}^2$ with lexicographical ordering and define $\hat{\nu} : \hat{K} :\to \hat{\Gamma}$ by

$$\begin{aligned} \nu(v) &= (0,1) \\ \nu(t) &= (1,0). \end{aligned}$$

Let ν denote the restriction of $\hat{\nu}$ to K. Then ν has value group $\mathbf{Z}$, so that $\mathrm{rk}\ \nu = \mathrm{rat.rk}\ \nu = 1$. It is not hard to see that $\mathrm{tr.deg}_k \nu = 0$, so that ν satisfies the strict inequality in (2.1). We have

$$\begin{aligned} \nu(v) = \nu(u) &= 1 \\ \nu(u + c_1 v) &= 2 \\ \nu(u + c_1 v + c_2 v^2) &= 3 \\ &\text{etc.} \end{aligned}$$

Geometrically, we are considering the algebroid curve in the plane Spec $k[u,v]$ defined by $t = 0$. Corresponding to the embedding of this curve into the plane we have an injective map $k[u,v] \to k[[v]]$ which sends u to $(-\sum_{i=1}^{\infty} c_i v^i)$. Then ν is the natural v-adic valuation of $k[[v]]$ pulled back to R.

Example 2.8. Let R and K be as above. Let $\Gamma = \mathbf{Q}$ and define a valuation ν with value group $\mathbf{Q}$ as follows.

$$\begin{aligned} \nu(v) &= 1 \\ \nu(u) &= 1\frac{1}{2} \\ \nu(u^2 + v^3) &= 3\frac{1}{3} \\ \nu((u^2 + v^3)^3 + v^{10}) &= 10\frac{1}{4} \\ &\text{etc.} \end{aligned}$$

Again, $\mathrm{rk}\ \nu = \mathrm{rat.rk}\ \nu = 1$ and $\mathrm{tr.deg}_k \nu = 0$, so we have a strict inequality in (2.1).

§3. FINITE GENERATION OF VALUATIONS AND LINEAR EQUIVALENCE OF TOPOLOGIES.

In the last two examples of §2 ν is not determined by its values on any finite subset of R. This fact motivates all the subsequent investigation: equality in (2.1) is equivalent to saying that ν is determined by a finite amount of data. Most of the present paper is devoted to the theory of valuations satisfying equality in (2.3). It is related to several different topics in commutative algebra and singularity theory, and we shall briefly discuss these connections here. First, we introduce some definitions and notation. Let

$$\Phi := \nu(R \setminus \{0\}) \subset \Gamma^+.$$

Φ is an additive semigroup containing 0. For $\alpha \in \Phi$, let

$$(3.1) \qquad \begin{aligned} p_\alpha &:= \{x \in R \mid \nu(x) \geq \alpha\} \\ p_{\alpha+} &:= \{x \in R \mid \nu(x) > \alpha\}. \end{aligned}$$

Associated to the pair (R, ν) we have two graded algebras:

$$\mathrm{Gr}_\nu R := \bigoplus_{\alpha \in \Phi} p_\alpha$$

$$\mathrm{gr}_\nu R := \bigoplus_{\alpha \in \Phi} \frac{p_\alpha}{p_{\alpha+}}.$$

Our intuitive idea of a valuation determined by a finite amount of data corresponds to the noetherian property of $\mathrm{gr}_\nu R$ and $\mathrm{Gr}_\nu R$. A good, if somewhat optimistic, model for the subsequent theory is the case when R is a regular local ring of dimension 2.

THEOREM 3.1. *[14, Theorem 8.6] Let R be a regular 2-dimensional local ring with field of fractions K and let ν be a valuation of K such that $R < R_\nu$. Assume that the residue field k of R is algebraically closed. Then equality holds for ν in (2.1) if and only if $\mathrm{Gr}_\nu R$ is finitely generated over R. If that is the case, any minimal set of generators of $\mathrm{gr}_\nu R$ over k, pulled back to R in an arbitrary way, has the form $Q_0, Q_1, \dots, Q_{g+1}$, where (Q_0, Q_1) are regular parameters for R and Q_{g+1} defines a plane curve singularity one of whose branches C has g Puiseux exponents. For $1 \leq i \leq g+1$, Q_i defines an analytically irreducible plane curve singularity with $i-1$ Puiseux exponents and having maximal contact with C.*

Theorem 3.1 may be viewed as a structure theorem for valuations centered in a regular 2-dimensional local ring with algebraically closed residue field. It can also be used to establish the "equivalence of categories" between valuations centered in R, plane curve singularities in Spec R, complete ideals in R and sandwiched surface singularities—the normal singularities of surfaces which birationally dominate Spec R. In particular, Theorem 3.1 contains the classical theory of plane curve singularities and maximal contact, done in greater generality: we need not assume that R is Henselian (we do not use the Weierstrass preparation theorem, nor the fact that char $k = 0$, nor even that $k \subset R$). See [14] and [16, Chapter II] for details and proofs.

Remark 3.2. A contraction of an ideal of R_ν to R is called a ν-**ideal.** By Theorem 1.7 (3) the set of ν-ideals in R is totally ordered by inclusion. All the p_α and $p_{\alpha+}$ are ν-ideals and the set $\{p_\alpha\}_{\alpha\in\Phi}$ gives the complete list of ν-ideals in R. The way to think of the statement that the natural images of $Q_0,\ldots,Q_{g+1}$ in $\mathrm{Gr}_\nu R$ generate $\mathrm{Gr}_\nu R$ as an R-algebra is that any ν-ideal p_α is generated by the set

$$\{\prod_{i=0}^{g+1} Q_i^{\gamma_i} \mid \sum_{i=0}^{g+1}\gamma_i \geq \alpha\}.$$

To give an analogue of Theorem 3.1 in higher dimensions we need a few more definitions.

DEFINITION 3.3. *We say that the semigroup Φ is* **archimedian** *if for any $\alpha,\beta\in\Phi$, $\alpha\neq 0$, there exists $r\in\mathbb{N}$ such that $r\alpha>\beta$.*

Let 1 denote the smallest non-zero element of Φ (such an element exists because R is noetherian). For $l\in\mathbb{N}$, let

$$l := l\cdot 1\in\Phi.$$

Thus we think of $\mathbb{N}$ as a subset of Φ. In particular, we may talk about the ν-ideals p_l for $l\in\mathbb{N}$. Note that by definition, $m^l\subset p_l$ for any $l\in\mathbb{N}$.

DEFINITION 3.4. *We say that the m-adic and the ν-adic topologies in R are* **linearly equivalent** *if there exists $r\in\mathbb{N}$ such that $p_{rl}\subset m^l$ for any $l\in\mathbb{N}$.*

DEFINITION 3.5. *Let $A=\oplus_{\alpha\in\Phi}A_\alpha$ be a Φ-graded k-algebra with $A_0=k$. We say that A is* **weakly noetherian** *of dimension d if the following two conditions hold:*

(1) *A contains d algebraically independent elements over k;*
(2) *there exists a polynomial $F(l)\in\mathbb{N}[l]$ such that for any $l\in\mathbb{N}$*

$$\sum_{\substack{\alpha\in\Phi\\ \alpha\leq l}} \dim_k A_\alpha \leq F(l).$$

This definition means that in terms of the growth of $\dim_k \alpha$, A behaves like a finitely generated k-algebra of transcendence degree d. In particular, if A is weakly noetherian of dimension d, we have $\dim\ A\leq d$. If, in addition, A is an integral domain (as will be the case in our applications) then the field of fractions of A is finitely generated over k. We can now state the main theorem.

THEOREM 3.6. *Let (R,m,k) be a local noetherian domain with field of fractions K. Let ν be a valuation of K centered in R. Then the following two conditions are equivalent:*

(1) *R is analytically irreducible (i.e. the m-adic completion $\hat{R}$ of R has no zero divisors), Φ is archimedian and*

$$\mathrm{rat.rk}\ \nu + \mathrm{tr.deg}_k\nu = \dim\ R.$$

(2) *$\mathrm{gr}_\nu R$ is weakly noetherian and the m-adic and the ν-adic topologies in R are linearly equivalent.*

Remark 3.7. Let $n = \text{rat.rk}\,\nu + \text{tr.deg}_k\nu$. It is not hard to show that for *any* ν centered in R we have $\text{rat.rk}\,\nu + \text{tr.deg}_k\nu = \text{tr.deg}_k \text{gr}_\nu R$. Hence, if $\text{gr}_\nu R$ is weakly noetherian, it is weakly noetherian of dimension n.

Remark 3.8. It is an easy exercise to prove that if the m-adic and the ν-adic topologies are equivalent (linearly or not) then Φ is archimedian and R is analytically irreducible. The only hard part of the theorem is to prove linear equivalence of topologies from (1).

Theorem 3.6 is proved by reducing to the case of divisorial valuations, which we state as a separate proposition because of its importance for the applications below.

PROPOSITION 3.9. *Let ν be a divisorial valuation centered in an analytically irreducible local noetherian domain (R, m, k). Then the m-adic and the ν-adic topologies in R are linearly equivalent.*

To motivate Proposition 3.9, we recall some old results of David Rees on ideal-adic topologies in R. Given an ideal I in any ring R, the I-adic order on R is defined by

$$I(f) := \max\{n \in \mathbb{N} \mid f \in I^n\} \qquad \text{for } f \in R.$$

A more invariant notion is the reduced order:

$$\bar{I}(f) := \lim_{n\to\infty} \frac{I(f^n)}{n}.$$

A priori, it is not obvious that $\bar{I}(f)$ is a rational number for every f, or even that it is finite. However, the following theorem of David Rees implies that $\bar{I}(f)$ is a rational number whose denominator is bounded uniformly for all $f \in R$. In the statement of the theorem below, the reader may either take R to be a domain or extend the definition of valuations to non-domains to be valuations on R modulo one of the minimal primes.

THEOREM 3.10 (DAVID REES [13]). *Let R be a noetherian ring and I an ideal. Then there exists a finite collection $\nu_1, \dots, \nu_r$ of divisorial valuations on R and positive integers $e_1, \dots, e_r$ such that for any $f \in R$*

$$\bar{I}(f) = \min_{1\le i\le r} \frac{\nu_i(f)}{e_i}.$$

$\bar{I}(f) = \infty$ if and only if f is nilpotent.

A geometric proof of David Rees's theorem may be obtained by first reducing to the case when R is a Nagata domain. Then we may consider the normalized blowing up $\pi : X \to \operatorname{Spec} R$ along I. Let $E_1, \dots, E_r$ be the irreducible components of $\pi^{-1}(m)$. We may take ν_i to be the divisorial valuation associated with E_i as in Example 1.5. Letting $e_i = \nu_i(I)$ (the minimum of ν_i on I), David Rees's theorem follows easily.

By Theorem 3.10, proving that any divisorial valuation ν centered in a local noetherian domain defines a topology linearly equivalent to the m-adic one is the same as proving that all the ν-adic topologies for divisorial ν are linearly equivalent to each other.

The rest of the paper is devoted to applications of Theorem 3.6 and related topics. The first one is really a lemma on the way to proving Theorem 3.6 (it is needed to reduce to the divisorial case).

LEMMA 3.11. *Let $(S, m_0) \subset (R, m)$ be a finite extension of two analytically irreducible local noetherian domains. Let $L \subset K$ be the fields of fractions of S and R, respectively. Then there exists $r \in \mathbb{N}$ such that for any $f \in R$*

$$m_0(N_{K/L}(f)) \leq rm(f).$$

The corollary of Proposition 3.9 needed to prove the theorems of the next section is

THEOREM 3.12. *Let (R, m) be an analytically irreducible local noetherian domain. Let $R \subset R'$ be a finitely generated R-algebra and let m' be any prime ideal of R' lying over m. Then the m-adic and the m'-adic topologies on R are linearly equivalent. That is, there exists $r \in \mathbb{N}$ such that for any $n \in \mathbb{N}$*

$$(m')^{rn} \cap R \subset m^n.$$

To prove Theorem 3.12, observe that any finitely generated extension can be obtained by composing three basic types of extensions: purely transcendental, finite and birational. In the purely transcendental case the theorem is trivial, the finite case is easy and the birational case follows from Proposition 3.9 and Theorem 3.10. In §5 we shall see that even if R' is not a finitely generated R-algebra, there is a class of situations (namely, when the Gabrielov rank condition is satisfied) in which one can guarantee linear equivalence of topologies.

§4. THE LINEAR ARTIN APPROXIMATION THEOREM.

First, we state some corollaries to Theorem 3.12 having to do with approximate (in the Krull topology) factorization of polynomials.

COROLLARY 4.1. *Let R be a noetherian ring and $T_1, \dots, T_n$ independent variables. Let m be a maximal ideal of R and $p \subset (m, T)$ a prime ideal of $R[[T]]$ such that $\frac{R[[T]]_{(m,T)}}{p}$ is analytically irreducible. Let $\bar{R}$ be a finitely generated extension of R and let*

$$p\bar{R}[[T]] = q_1 \cap \cdots \cap q_s \tag{4.1}$$

be a primary decomposition of p in $\bar{R}[[T]]$. Let $p_i := \sqrt{q_i}$. Let $\bar{m} \subset \bar{R}[[T]]$ be a prime ideal such that $\bar{m} \cap R[[T]] = (m, T)$. $\bar{m}$ must contain one of the p_i. Assume that $p_1 \subset \bar{m}$. Then there exists $r \in \mathbb{N}$ such that for any $n \in \mathbb{N}$

$$(\bar{m}^{rn} + p_1) \cap R[[T]] \subset (m, T)^n + p.$$

Remark 4.2. This is a special case of the linear Artin approximation theorem, stated below. In particular, we can apply this corollary to the case when R is a UFD, p is generated by one power series $F \in R[[T]]$, and (4.1) corresponds to a factorization F in $\bar{R}[[T]]$. Say, $F = F_1 F_2$ in $\bar{R}[[T]]$. Then Corollary 4.1 says that there exists $r \in \mathbf{N}$ such that if

$$\tilde{F} \equiv F_1 \tilde{F}_2 \mod \bar{m}^{rn},$$

where $\tilde{F}_2 \in \bar{R}[[T]]$, $\tilde{F} \in R[[T]]$, then

$$\tilde{F}_2 \cong F_2 g \mod \bar{m}^n \qquad \text{and}$$
$$\tilde{F} \cong F g \mod \bar{m}^n$$

for some $g \in R[[T]]$. In other words, approximate factorization of an element of $R[[T]]$ in $\bar{R}[[T]]$ is close to the actual factorization, and the estimate is linear in n.

Remark 4.3. Corollary 4.1 can be strengthened as follows. Let R be a noetherian ring, $T_1, \ldots, T_n$ independent variables and A a noetherian ring such that

$$R[T] \subset A \subset R[[T]].$$

Let m be a maximal ideal of R and p a prime ideal of A such that $p \subset m$ and $\frac{A_m}{p}$ is analytically irreducible. Let $\bar{R}$ be a finitely generated R-algebra and let B be a noetherian $\bar{R}$-algebra such that

$$A \otimes_R \bar{R} \subset B \subset \bar{R}[[T]].$$

Assume that (m, T) is a maximal ideal of A and that both $A \otimes_R \bar{R}$ and B have $\bar{R}[[T]]$ as their T-adic completion. Let $\{p_i\}_{1 \leq i \leq s}$ be the associated primes of p in B. Let $\bar{m}$ be any prime ideal of $\bar{R}$ lying over (m, T). Assume that $p_1 \subset \bar{m}$. Then there exists $r \in \mathbf{N}$ such that for any $n \in \mathbf{N}$

$$(\bar{m}^{rn} + p_1) \cap R[[T]] \subset (m, T)^n + p.$$

The next Corollary has to do with the notion of *superficial element* (cf. [26, Chapter VIII, §8, p. 285]).

COROLLARY 4.4. *Let R be a noetherian ring without nilpotents and I an ideal of R. Let x be an element of R. Assume that for any minimal prime p of R, such that $I + p \neq R$, we have $x \notin p$. Assume also that the degree over $\mathrm{Gr}_I R$ of any element of the integral closure of $\mathrm{Gr}_I R$ in its total ring of fractions is uniformly bounded (this condition always holds when R is Nagata). Then there exists $r \in \mathbf{N}$ such that for any $k, n \in \mathbf{N}$*

$$I^k : x^n \subset I^{k-rn}$$

(here we adopt the convention that $I^n = R$ if $n \leq 0$).

Now we state the linear Artin approximation theorem.

THEOREM 4.5. *Let (R, m, k) be one of the following:*
(1) *an excellent Henselian local ring containing* $\mathbb{Q}$
(2) *the henselization of the ring* $k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ *where k is a field or an excellent Dedekind domain*
(3) *a convergent power series ring in the sense of [11, §45]*
(4) *a complete local ring.*

Consider a system of algebraic (resp. analytic in (3), resp. formal in (4)) equations over R. By a system of equations we mean a finitely generated R-algebra of the form $\frac{R[T_1,\dots,T_n]}{I}$ *(resp.* $\frac{R\{T_1,\dots,T_n\}}{I}$*, resp.* $\frac{R[[T_1,\dots,T_n]]}{I}$*), where I is a radical ideal such that $I \cap R = 0$. Then there exists $r \in \mathbb{N}$ such that for any $l \in \mathbb{N}$ and any $t'_1, \dots, t'_n \in R$ such that*

$$I \subset (T_1 - t'_1, \dots, T_n - t'_n) + m^{rl}$$

there exist $t_1, \dots, t_n \in R$ such that

$$I \subset (T_1 - t_1, \dots, T_n - t_n) \qquad \text{and} \qquad t_i - t'_i \in m^l$$

for $1 \leq i \leq n$. In other words, any approximate solution can be linearly approximated by an exact solution.

In [7] this theorem is proved for isolated hypersurface singularities. It is shown that in that case r may be taken to be the multiplicity of the singularity times its Milnor number.

Example 4.6. Many of the results stated here were motivated by Izumi's "nearness to p-powers" [3, §5]. Let R be as in Theorem 4.5. Let $p \in \mathbb{N}$ and consider $u \in R \backslash K^p$, where K^p denotes the set of p-th powers of elements of the field of fractions K of R. Then there exists $r \in \mathbb{N}$ such that for any $t_1, t_2 \in R$ and any $l \in \mathbb{N}$ with $t_1^p - ut_2^p \in m^{rl}$, we have $t_1, t_2 \in m^l$.

§5. CONVERGENCE OF FORMAL FUNCTIONS.

Finally, we discuss the following problem of M. Artin and A. Grothendieck, solved by Gabrielov in 1974. Let $\phi : (R, m) \hookrightarrow (R', m')$ be an injective local homomorphism of complex-analytic local rings. That is, each of R and R' is the quotient of a convergent power series ring of the form $\mathbb{C}\{x_1, \dots, x_n\}$ by an ideal. Consider the induced homomorphism $\hat{\phi} : \hat{R} \to \hat{R}'$ on the formal completions of R and R'. The question is: is $\hat{\phi}$ injective?

Gabrielov gave an example showing that $\hat{\phi}$ need not be injective in general. He also gave a sufficient condition, now known as the Gabrielov's rank condition, for $\hat{\phi}$ to be injective.

Example 5.1. Recall the example of Osgood of an injective local homomorphism from $\mathbb{C}\{x_1, x_2, x_3\}$ into $\mathbb{C}\{y_1, y_2\}$:

$$\phi(x_1) = y_1$$
$$\phi(x_2) = y_1 y_2$$
$$\phi(x_3) = y_1 e^{y_2} = y_1 + y_1 y_2 + \frac{y_1 y_2^2}{2!} + \dots$$

Consider the following sequence of functions in $\mathbf{C}[x_1, x_2, x_3]$.

$$
\begin{aligned}
f_1 &:= x_3 - x_1 - x_2 \sim \frac{y_1 y_2^2}{2!} + \dots \\
f_2 &:= x_1(x_3 - x_1 - x_2) - \frac{x_2^2}{2!} \sim \frac{y_1^2 y_2^3}{3!} + \dots \\
&\dots
\end{aligned}
$$

For a power series $f \in \mathbf{C}\{x_1, \dots, x_n\}$, $f = \sum_\alpha c_\alpha x^\alpha$, let size$(f)$ denote

$$\text{size}(f) := \max\{|c_\alpha| \mid c_\alpha \neq 0\},$$

where we allow size to be infinite. Then size$(f_i) = 1$ for all $i \in \mathbf{N}$, while size$(\phi(f_i))$ goes rapidly to zero as $i \to \infty$. Hence, there exist c_i, $i \in \mathbf{N}$ such that

$$x_4 := \sum_{i=1}^{\infty} c_i f_i$$

is divergent as a power series in x but convergent as a power series in y. Extending ϕ to x_4 in an obvious way, we get an example of an injective local analytic map $\phi : \mathbf{C}\{x_1, x_2, x_3, x_4\} \to \mathbf{C}\{y_1, y_2\}$ such that $\hat{\phi}$ is not injective. That is, we produced four convergent power series such that there are no analytic relations between them, but there is a formal relation.

Next, we state Gabrielov's rank condition. Assume for simplicity that R' is a domain. Write

$$
\begin{aligned}
R &= \frac{\mathbf{C}\{x_1, \dots, x_n\}}{I} \\
R' &= \frac{\mathbf{C}\{y_1, \dots, y_t\}}{J}
\end{aligned}
\tag{5.1}
$$

Then the rank of ϕ is defined to be

$$\text{rk } \phi := \text{rk } \left(\frac{\partial x_i}{\partial y_j}\right)_{1 \leq j \leq t}^{1 \leq i \leq n},$$

where rank means rank over the the field of fractions of R'. To use Gabrielov's notation, we put

$$
\begin{aligned}
r_1 &:= \text{rk } \phi \\
r_2 &:= \dim \ \hat{\phi}(\hat{R}) \\
r_3 &:= \dim \ R.
\end{aligned}
$$

It is easy to see that

$$r_1 \leq r_2 \leq r_3. \tag{5.2}$$

The condition $r_1 = r_2$ is called Gabrielov's rank condition. Gabrielov's famous theorem is

THEOREM 5.2 [2]. *If* $r_1 = r_2$ *then* $r_1 = r_3$.

The above equality implies, in particular, that ϕ is injective. The converse is not true: consider Example 5.1. We have $r_1 = 2$, $r_2 = 3$, $r_3 = 4$. The restriction of ϕ to $\mathbf{C}\{x_1, x_2, x_3\}$ is injective, the induced map $\mathbf{C}[[x_1, x_2, x_3]] \to \mathbf{C}[[y_1, y_2]]$ is injective but it does not satisfy Gabrielov's rank condition.

Valuation-theoretic ideas discussed in this paper provide a relatively simple proof of Gabrielov's theorem. This proof is purely algebraic in nature and can be generalized to a wider class of rings than the complex-analytic rings. In order to state Gabrielov's theorem we need the notion of rank of ϕ. To define that in a general context, we need to know that our ring behaves in a reasonable way with respect to derivations and differentials. In particular, there arises the problem of finding a good substitute for Kähler differentials: the usual Kähler differentials, even for very reasonable local rings, are too large to deal with. More precisely, we would like to have a notion of differentials such that the module of differentials of a formal or convergent power series ring R over a field k in n variables is a free R-module of rank n. We therefore propose the notion of separated Kähler differentials, which should play the analogous role for the study of local rings as the usual Kähler differentials do in the global study of algebraic varieties.

DEFINITION 5.3. *Let R be a ring, I an ideal and M an R-module. We say that M is I-***adically regular** *if for any finitely generated submodule M_0 of M,*

$$\bigcap_{i=0}^{\infty}(M_0 + I^i M) = M_0.$$

For example, if R is a local ring with maximal ideal I then any finitely generated R-module is I-adically regular.

Now let (R, m, k) be a local noetherian ring containing a field. Then R contains a quasi-coefficient field k_0 (that is, k is separable algebraic over k_0). Consider the module Ω_{R/k_0} of Kähler differentials of R over k_0.

DEFINITION 5.4. *The* **separated Kähler differentials,** *denoted $\bar{\Omega}_{R/k_0}$, are defined by*

$$\bar{\Omega}_{R/k_0} := \frac{\Omega_{R/k_0}}{\bigcap_{i=0}^{\infty} m^i \Omega_{R/k_0}}.$$

DEFINITION 5.5. *We say that R is* **formally Jacobian** *if $\bar{\Omega}_{R/k_0}$ is m-adically separated.*

Remark 5.6. The canonical derivation $d : R \to \Omega_{R/k_0}$ induces a derivation $\bar{d} : R \to \bar{\Omega}_{R/k_0}$. If R is formally Jacobian, $\bar{d}$ is characterized by the following universal property. For any m-adically regular R-module M and any derivation $d' : R \to M$ vanishing on k_0, there exists a unique homomorphism

$$f : \bar{\Omega}_{R/k_0} \to M$$

such that $d' = f \circ \bar{d}$. In other words, for an m-adically regular R-module M

$$\mathrm{Der}_{k_0}(R, M) \cong \mathrm{Hom}_R(\bar{\Omega}_{R/k_0}, M).$$

One also has the analogues of the first and second fundamental exact sequences.

Example 5.7. Let $R = k[[x_1, \dots, x_n]]$ (resp. $k\{x_1, \dots, x_n\}$ when k is equipped with a multiplicative valuation in the sense of [11, §45]). Then R is formally Jacobian and

$$\tilde{\Omega}_{R/k_0} = \bigoplus_{i=1}^{n} R dx_i.$$

If char $k = 0$, any localization of the ring of the form $k\{x_1, \dots, x_n\}[[y_1, \dots, y_l]][z_1, \dots, z_t]$ at a maximal ideal is formally Jacobian.

We give two equivalent characterizations of formally Jacobian rings.

THEOREM 5.8. *Let (R, m, k) be a local noetherian ring, containing a field. Let k_0 be a quasi-coefficient field of R and let $\hat{R}$ denote the m-adic completion. Then the following conditions are equivalent.*

(1) *R is formally Jacobian*
(2) *$\tilde{\Omega}_{R/k_0}$ is a finite R-module*
(3)

$$\tilde{\Omega}_{\hat{R}/k} \cong \tilde{\Omega}_{R/k_0} \otimes_R \hat{R}.$$

Let R be an n-dimensional regular local ring. Then R is formally Jacobian if and only if $\mathrm{Der}_{k_0}(R)$ is a free R-module of rank n. In particular, when char $k_0 = 0$, R is formally Jacobian if and only if the weak Jacobian condition (WJ) of [8] holds in R.

Now let $\phi : (R, m, k) \to (R', m', k')$ be an injective local homomorphism of formally Jacobian local rings. Assume that $\mathbb{Q} \subset R$, that R' is an analytically irreducible domain and that k' is algebraic over k. Then we may use the same quasi-coefficient field k_0 for R and for R'. We have a natural map

$$\overline{d\phi} : \tilde{\Omega}_{R/k_0} \otimes_R R' \to \tilde{\Omega}_{R'/k_0}.$$

We define

$$r_1 := \mathrm{rk}\ \phi = \mathrm{rk}(\mathrm{Im}(\overline{d\phi})),$$

where rank means the dimension as a vector space after tensoring with the field of fractions of R'. (5.2) holds, as before. We now state the more general "Gabrielov's theorems" (many of the formulations given below come from the papers of Izumi [3–6]).

THEOREM 5.9. *The following conditions are equivalent.*

(1) $r_1 = r_2$
(2) $r_1 = r_3$
(3) *the m-adic and the m'-adic topologies on R are linearly equivalent.*

Assume, furthermore, that R and R' are complex-analytic local rings. Say, R and R' are given by (5.1). Choose $\epsilon > 0$ such that the power series $\phi(x_i)$ converge in the neighbourhood of the polydisc $|y_i| \le \epsilon$.

THEOREM 5.10. *Each of the equivalent conditions of Theorem 5.9 is also equivalent to each of the following.*

(1) ϕ *is a closed embedding in the Krull topology*
(2) $\mathrm{Im}(\phi)$ *is closed in* R' *in the Krull (i.e. the* m'*-adic) topology*
(3) $(\hat{\phi})^{-1}(R') = R$
(4) *there exists a positive constant* $\beta \in \mathbf{R}$ *such that for any* $f \in \mathbf{C}[x_1, \ldots, x_n] \cap (m^l \setminus m^{l+1})$,

$$\mathrm{size}(f) \leq \beta^l \max_{|y| \leq \epsilon} |\phi(f)(y)|.$$

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